ON THE OPTIMIZATION OF METHODS FOR SOLVING CERTAIN PROBLEMS OF APPROXIMATE ANALYSIS

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Abstract

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ON THE OPTIMIZATION OF METHODS FOR SOLVING CERTAIN PROBLEMS OF APPROXIMATE ANALYSIS

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II. 1. In this note we consider questions of optimizing algorithms for solving three problems of approximate analysis: numerical differentiation, numerical integration, and interpolation. Let Ω be a domain in Euclidean space of any finite dimension n, let $\varphi(x)$ be a function from some given class Φ, and suppose it is required to construct optimal, in a certain sense, formulas of numerical differentiation and integration of the form

$$D^{\alpha}\varphi\left(x^{(k_p)}\right) \approx \sum_{i=1}^{N} c_{k_i}\varphi\left(x^{(k_i)}\right) \tag{1}$$

(α is a multi-index with integer nonnegative components

$$\alpha = (\alpha_1, \dots, \alpha_n), \qquad |\alpha| = \sum_{i=1}^{n} \alpha_i),$$

$$\int_{\Omega} \varphi(x)\,dx \approx \sum_{i=1}^{N} c_{l_i}\varphi\left(x^{(l_i)}\right). \tag{2}$$

This question has recently been intensively developed in various aspects (see ([3–9])). In particular, in the fundamental works of S. L. Sobolev ([3]) and S. M. Nikol' skii ([4]) a functional-theoretic principle was developed for constructing formulas of numerical integration, which can be applied equally well to the construction of formulas of numerical differentiation.

The point is the following. Considering formulas (1) and (2) on some class of functions Φ, let us associate with them, respectively, the error functionals

$$(k, \varphi) = D^{\alpha}\varphi\left(x^{(k_p)}\right) - \sum_{i=1}^{N} c_{k_i}\varphi\left(x^{(k_i)}\right), \tag{3}$$

$$(l, \varphi) = \int_{\Omega} \varphi(x)\, dx - \sum_{i=1}^{N} c_{l_i}\varphi\left(x^{(l_i)}\right), \tag{4}$$

and suppose that the functional class under study forms a Banach space, and moreover such that the functionals (3) and (4), defined on this space, are continuous. In this case it is natural to estimate the quality of the formulas by means of the norms of these functionals

$$\|k\|_{\Phi^*} = \sup_{\varphi \neq 0} |(k, \varphi)| / \|\varphi\|_{\Phi} = F_k\left(c_{k_1}, \dots, c_{k_N}, x^{(k_1)}, \dots, x^{(k_N)}\right), \tag{5}$$

$$\|l\|_{\Phi^*} = \sup_{\varphi \neq 0} |(l, \varphi)| / \|\varphi\|_{\Phi} = F_l\left(c_{l_1}, \dots, c_{l_N}, x^{(l_1)}, \dots, x^{(l_N)}\right), \tag{6}$$

The search for the minimum of the functions F_k and F_l is a typical minimax problem. The c_{k_i} and $x^{(k_i)}$, $i = 1, 2, \dots, N$, that realize the minimum of the function F_k give an optimal formula of numerical differentiation, while the c_{l_i} and $x^{(l_i)}$, $i = 1, 2, \dots, N$, that realize the minimum of the function F_l give an optimal formula of numerical integration.

In an analogous way the problem of finding an optimal interpolation formula can be posed. The problem of interpolation

is closely connected with the two preceding problems of approximate analysis (in the classical theory, formulas of numerical differentiation and integration are obtained respectively by differentiating and integrating interpolation formulas). Let us recall its formulation. For an approximate representation of a function $\varphi(x)$ by elements of some finite set $\varphi_1(x), \varphi_2(x), \dots, \varphi_M(x)$, one uses the values that this function assumes on some finite set of points $x^{(m_i)}$, $i = 1, 2, \dots, N$. Here we shall study linear interpolation of the form

$$\varphi(x) \approx \sum_{i=1}^{M} a_i \varphi_i(x). \tag{7}$$

The coefficients a_i are found from the condition that the two sides of equality (7) coincide at the points $x^{(m_i)}$, called interpolation nodes. As a result we obtain a system of N equations with M unknowns. Suppose that $M = N$ and that the determinant of this system is nonzero. In this case the interpolation problem is uniquely solvable, and the problem will be classical. After simple manipulations, formula (7) can be transformed to the form

$$\varphi(x) \approx \sum_{i=1}^{N} c_{m_i}(x)\varphi\left(x^{(m_i)}\right); \tag{8}$$

the $c_{m_i}(x)$ do not depend on $\varphi(x)$ and are determined entirely by the functions $\varphi_i(x)$ and the interpolation nodes. Let us also note that, since

$$\varphi\left(x^{(m_j)}\right) = \sum_{i=1}^{N} c_{m_i}\left(x^{(m_j)}\right)\varphi\left(x^{(m_i)}\right),$$

it follows that

$$c_{m_i}\left(x^{(m_j)}\right) = 1, \quad \text{if } i = j; \qquad c_{m_i}\left(x^{(m_j)}\right) = 0, \quad \text{if } i \neq j.$$

In what follows we shall seek interpolation formulas directly in the form (8).

The error of the interpolation formula (8) at some point z is a functional on φ,

$$(m, \varphi) = \varphi(z) - \sum_{i=1}^{N} c_{m_i}(z)\varphi\left(x^{(m_i)}\right). \tag{9}$$

Therefore, if on the functional class Φ under consideration it is continuous, then the quality of the interpolation formula can be characterized by its norm

$$\|m\|_{\Phi^*} = \sup_{\varphi \neq 0} |(m, \varphi)| / \|\varphi\|_{\Phi} = F_m\left(c_{m_1}(z), \dots, c_{m_N}(z), x^{(m_1)}, \dots, x^{(m_N)}\right).$$

The $c_{m_i}(z)$ and $x^{(m_i)}$ that realize the minimum of the function F_m will give the optimal interpolation formula.

The complete problem of minimizing the functions F_k, F_l, F_m is extremely difficult. Therefore we shall restrict ourselves to considering its simplified variant. Namely, for a fixed system of nodes we shall find formulas optimal with respect to the coefficients. In this case the problem of interest to us—finding optimal formulas—admits a comparatively simple solution.

2. We shall construct optimal formulas for the class of spaces $H_2^{(\mu)}(\Omega)$, which are a generalization of the spaces of S. L. Sobolev widely known in analysis. Let us give the necessary definitions.

Let E_n be the n-dimensional Euclidean space of points $x = (x_1, \dots, x_n)$, and let E^n be the n-dimensional Euclidean space of points $\xi = (\xi^1, \dots, \xi^n)$. We shall regard the variables x and ξ as dual with respect to the bilinear

linear form $\langle x\xi\rangle = x_1\xi^1+...+x_n\xi^n$. Let Ω be a bounded domain in the space E_n. It will be more convenient for us to define the basic space $H_2^{(\mu)}(\Omega)$ in terms of generalized functions (see the monograph ([10])). Denote by $H_2^{(\mu)}(E_n)$ the space of generalized functions u whose Fourier transforms $u(\xi)$ are square-integrable with weight $\mu^2(\xi) > 0$. We shall define the space $H_2^{(\mu)}(\Omega)$ as the space whose elements are restrictions of functions from the space $H_2^{(\mu)}(E_n)$ to the bounded domain $\Omega \subset E_n$. We introduce the topology in these spaces as follows:

$$\|u\|_{H_2^{(\mu)}(E_n)} = \left(\int_{E^n} |\hat{u}(\xi)|^2\mu^2(\xi)\,d\xi\right)^{1/2},$$

$$\|u\|_{H_2^{(\mu)}(\Omega)} = \inf_{u^c}\|u^c\|_{H_2^{(\mu)}(E_n)},$$

where the lower bound is taken over all possible extensions. We introduce one more space, $\dot{H}_2^{(\mu)}(\Omega)$, closely connected with the space $H_2^{(\mu)}(\Omega)$. We define it as the closure of the set $C_0^\infty(\Omega)$ in the metric $H_2^{(\mu)}(E_n)$. As is not difficult to show (see ([11])), $H_2^{(\mu)*}(\Omega) = \dot{H}_2^{(1/\mu)}(\Omega)$. We shall assume the space $H_2^{(\mu)}(\Omega)$ to be embedded in the space of ordinary continuous functions. The embedding condition has the form

$$\int_{E^n} \frac{1}{\mu^2(\xi)}\,d\xi < \infty \tag{10}$$

(see ([11,12])). The space $H_2^{(\mu)*}(\Omega)$ is isomorphic and isometric to the space L_2 and, consequently, is an ordinary Hilbert space in which an inner product is introduced in the natural way. Suppose now that in the problems of interpolation and numerical integration $\varphi(x) \in H_2^{(\mu)}(\Omega)$, and in the problem of numerical differentiation $D^\alpha\varphi(x) \in H_2^{(\mu)}(\Omega)$, and that the embedding condition (10) is satisfied. In this case the error functionals (3), (4), and (9) will be continuous, and the theoretical-functional formulation of the optimization problem makes sense. The functionals (3), (4), and (9) in the notation of generalized functions are written, respectively, in the form

$$k(x) = (-1)^{|\alpha|}D^\alpha\delta(x - x^{(k_p)}) - \sum_{i=1}^{N} c_{k_i}\delta(x - x^{(k_i)}),$$

$$l(x) = \mathcal{E}_\Omega(x) - \sum_{i=1}^{N} c_{l_i}\delta(x - x^{(l_i)})$$

($\mathcal{E}_\Omega(x)$ is the characteristic function of the domain Ω),

$$m(x) = \delta(x - z) - \sum_{i=1}^{N} c_{m_i}(z)\delta(x - x^{(m_i)}).$$

If we assume that the nodes in all formulas are fixed and distinct, then the problem of constructing optimal formulas essentially reduces to the best approximation of the functionals $(-1)^{|\alpha|}D^\alpha\delta(x - x^{(k_p)})$, $\mathcal{E}_\Omega(x)$, and $\delta(x - z)$ by linear combinations of δ-functions; that is, within the framework of the theory of generalized functions it is posed as a problem of best approximation in the Hilbert space $H_2^{(\mu)*}(\Omega)$, and we may use the usual technique in such cases. Namely, the coefficients in the optimal formulas are to be found from the orthogonality conditions.

$$\left(k(x), \delta\left(x - x^{(k_j)}\right)\right)_{H_2^{(\mu)*}(\Omega)} = 0, \quad j = 1, 2, \ldots, N,$$

$$\left(l(x), \delta\left(x - x^{(l_j)}\right)\right)_{H_2^{(\mu)*}(\Omega)} = 0, \quad j = 1, 2, \ldots, N,$$

$$\left(m(x), \delta\left(x - x^{(m_j)}\right)\right)_{H_2^{(\mu)*}(\Omega)} = 0, \quad j = 1, 2, \ldots, N.$$

As a result, to determine them we obtain the linear systems

$$\sum_{i=1}^{N} c_{k_i} v\left(x^{(k_j)} - x^{(k_i)}\right) = v_\alpha\left(x^{(k_j)} - x^{(k_p)}\right), \quad j = 1, 2, \ldots, N; \tag{11}$$

$$\sum_{i=1}^{N} c_{l_i} v\left(x^{(k_j)} - x^{(k_i)}\right) = g\left(x^{(l_j)}\right), \quad j = 1, 2, \ldots, N; \tag{12}$$

$$\sum_{i=1}^{N} c_{m_i} v\left(x^{(m_j)} - x^{(m_i)}\right) = v\left(x^{(m_j)} - z\right), \quad j = 1, 2, \ldots, N. \tag{13}$$

Here the notation has been introduced $v(x) = F^{-1}[\mu^{-2}(\xi)]$, $v_\alpha(x) = F^{-1}[(-1)^\alpha (i\xi)^\alpha \times \mu^{-2}(\xi)]$, $g(x) = \mathcal{E}_\Omega(x) * v(x)$, and F^{-1} is the inverse Fourier operator. Since all determinants of the systems are Gram determinants of linearly independent δ-functions and, consequently, are nonzero, these systems are uniquely solvable.

For the squares of the norms of the error functionals of the optimal formulas (we denote such functionals by zero indices) it is easy to obtain the expressions

$$\left\|k^0\right\|^2_{H_2^{(\mu)*}(\Omega)} = \frac{G\left(\delta\left(x - x^{(k_1)}\right), \ldots, \delta\left(x - x^{(k_N)}\right), (-1)^{|\alpha|} D^\alpha \delta\left(x - x^{(k_p)}\right)\right)}{G\left(\delta\left(x - x^{(k_1)}\right), \ldots, \delta\left(x - x^{(k_N)}\right)\right)}; \tag{14}$$

$$\left\|l^0\right\|^2_{H_2^{(\mu)*}(\Omega)} = \frac{G\left(\delta\left(x - x^{(l_1)}\right), \ldots, \delta\left(x - x^{(l_N)}\right), \mathcal{E}_\Omega(x)\right)}{G\left(\delta\left(x - x^{(l_1)}\right), \ldots, \delta\left(x - x^{(l_N)}\right)\right)}; \tag{15}$$

$$\left\|m^0\right\|^2_{H_2^{(\mu)*}(\Omega)} = \frac{G\left(\delta\left(x - x^{(m_1)}\right), \ldots, \delta\left(x - x^{(m_N)}\right), \delta(x - z)\right)}{G\left(\delta\left(x - x^{(m_1)}\right), \ldots, \delta\left(x - x^{(m_N)}\right)\right)}. \tag{16}$$

Remark. Although from the theoretical point of view the problem of constructing formulas optimal with respect to the coefficients may be regarded as solved, the practical solution of systems (11)—(13) is associated with considerable difficulties. Therefore the problem of finding simpler formulas, though not optimal, but in a certain sense close to them, is of interest. Numerical integration formulas possessing this property were constructed in works (13–14). These formulas proved to be close to optimal even in an entire class of spaces and, in this sense, are, by the definition of I. Babushka, universal.

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Note: Figure translations are in progress. See original paper for figures.