APPROXIMATION OF FUNCTIONS DEFINED IN A CONVEX POLYHEDRON

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Abstract

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MATHEMATICS

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APPROXIMATION OF FUNCTIONS DEFINED IN A CONVEX POLYHEDRON

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1. Let Γ be a convex bounded polyhedron in s-dimensional space $\mathbb{R}^s$; denote by $\rho(x,\Gamma)$ the distance from the point x to the nearest vertex of Γ. Further, let $\omega(\tau) = \omega_k(f;\tau;\Gamma)$, k be the modulus of continuity of the function f, defined on Γ; this means that

$$\omega(\tau) = \sup_{|y|\le\tau} \|\Delta_y^k f\|_C, \tag{1}$$

where the C-norm is taken over the domain of definition of the k-th difference of the function f.

Denote by $p_n(x)$ a polynomial of degree n in $x = (x_1, \dots, x_s)$, and let

$$\lambda_n(x) = \lambda_n(x,\Gamma) = \sqrt{\rho(x,\Gamma)} + n^{-1}. \tag{2}$$

Theorem 1. If $f \in C(\Gamma)$ and k is a fixed natural number, then there exists a sequence of polynomials $p_n(x)$, $n \ge k-1$, such that for $x \in \Gamma$

$$|f(x) - p_n(x)| \le A\omega(\lambda_n(x)/n). \tag{3}$$

Here $A = A(k,\Gamma)$, i.e., it depends only on k and Γ.

Thus Theorem 1 asserts the possibility of improving the approximation at the vertices of Γ. For functions of one variable, from (3) we obtain the known theorems of A. F. Timan ([1]) ($k = 1$) and V. K. Dzyadyk ([2])–G. Freud ([3]) ($k = 2$); for arbitrary k, Theorem 1 in the case of functions of one variable was proved by the author ([4],[5]). We note that the phenomenon of improvement of approximation at the endpoints of an interval was first discovered in ([6]).

2. Suppose that $f \in C^l(\Gamma)$ and

$$\sup_{|\alpha|=l} \omega_k(D^\alpha f; \tau; \Gamma) \leq \omega(\tau). \tag{4}$$

Corollary 1. There exists a sequence of polynomials $p_n(x)$, $n \geq k+l-1$, such that for $x \in \Gamma$

$$|f(x) - p_n(x)| \leq A\{\lambda_n(x)/n\}^l \omega(\lambda_n(x)/n). \tag{5}$$

Here $A = A(k+l, \Gamma)$.

Further, let U be a closed convex bounded subset of $\mathbb{R}^s$, containing at least one interior point. We shall call a point $x \in U$ conical if at this point there exist at least s linearly independent supporting hyperplanes. Suppose that U contains a finite number of conical points, and let $\rho(x)$ be the distance from x to the nearest conical point of U; define $\lambda_n(x)$ by equality (2). Under these assumptions we have

Corollary 2. If $f \in C^l(U)$ and for $k = 1$ condition (4) is satisfied, then there exists a sequence of polynomials $p_n(x)$, $n \geq l$, such that for $x \in U$

$$|f(x) - p_n(x)| \leq A\{\lambda_n(x)/n\}^l \omega(\lambda_n(x)/n). \tag{6}$$

Here $A = A(l, U)$.

3. The question arises of the possibility of improving the approximation at nonconical points of a convex set. In this direction the following negative result holds.

Theorem 2. *If $x_0 \in U$ is not a conical point, then there exists a function $f_0 \in C(U)$, satisfying a Lipschitz condition on U, such that the approximation*

$$|f_0(x) - p_n(x)| \leq \frac{A}{n}\left(\sqrt{|x - x_0|} + n^{-1}\right)$$

is impossible for any sequence of polynomials $p_n(x)$, $n = n_0, n_0 + 1, \ldots$, and for any constant $A = A(U)$.

The corresponding negative result for the one-dimensional case was obtained by I. E. Gopengauz ([7]).

4. Let us outline the proof of Theorem 1.

Lemma 1. *For every natural number k and integer $n \geq k - 1$ there exists a linear operator $P = P(n, k)$, mapping $C(0, 1)$ into the space of polynomials of degree n, such that:*

1) $\|P\| \leq A = A(k)$;
2) for $0 \leq x \leq 1$

$$|f(x) - P(f;x)| \le B(k)\omega_k\left(f; \sqrt{x(1-x)}/n + 1/n^2\right);$$

3) if $0 < a < a + \delta < 1$, then there exists $C = C(k, \delta)$ such that

$$\max_{a+\delta \le x \le 1} |f(x) - P(f;x)| \le C\left\{\max_{a \le x \le 1} |f(x)| + \omega_k(f; n^{-2})\right\}.$$

For the proof see ([5]).

Let now $Q \subset \mathbb{R}^s$ be a cube and let U be an open convex body which contains only one vertex of Q and intersects only those faces of Q that are adjacent to this vertex. Denote further by $U(\delta)$ the δ-expansion of U.

Lemma 2. *Let $f \in C(Q)$ and $f = 0$ outside $Q \cap U$; let $\omega(\tau)$ be (1) and $\lambda_n(x) = \lambda_n(x;Q)$. Then there exists a sequence of polynomials $p_n(x)$, $n = sj$, $j \ge k - 1$, such that:*

1) there exists a constant $A = A(k, Q)$ for which

$$|f(x) - p_n(x)| \le A\omega(\lambda_n(x)/n);$$

2) for a given $\delta > 0$ there exists $B = B(k, \delta, Q)$ such that

$$\max_{Q \setminus U(\delta)} |p_n(x)| \le B\omega(n^{-2}).$$

Proof. Without loss of generality we assume $Q = \{x \mid 0 \le x_i \le 1\}$; denote by P_i the operator $P(n, k)$ of Lemma 1 applied to functions $f(x) \in C(Q)$ with respect to the variable x_i, and let $P = P_1 P_2 \dots P_n$. Then $P(f;x)$ is a polynomial of degree sn, and the application of properties 1)–3) of the operator $P(n, k)$, established in Lemma 1, completes the proof.

Let now Π be a convex s-dimensional pyramid with vertex x_0; let U be an open convex body containing x_0 and not containing points of the base of Π, and let $U(\delta)$ be the δ-expansion of U.

Lemma 3. *If $f \in C(\Pi)$ and $f = 0$ outside $\Pi \cap U$, then there exists a sequence of polynomials $p_n(x)$, $n = js$, $j \ge k - 1$, such that:*

1) if $\omega(\tau)$ is defined by formula (1), then for some $A = A(k, \Pi)$

$$|f(x) - p_n(x)| \le A\omega\left(\left(\sqrt{|x - x_0|} + n^{-1}\right)/n\right);$$

2) for a given $\delta > 0$ there exists $B = B(k, \delta, \Pi)$ such that

$$\max_{\Pi \setminus U(\delta)} |p_n(x)| \le B\omega(n^{-2}).$$

Proof. Without loss of generality we assume $x_0 = 0$ and let Π_+ be the infinite pyramid generated by Π, and let $l_j(x)$, $j = 1, 2, \dots, N$, be the linear forms defining Π_+, i.e.,

$$\Pi_+ = \{x \in \mathbf{R}^s \mid l_j(x) \geq 0\}.$$

Define a mapping $A : \Pi_+ \to \mathbf{R}_+^N$ by the formula

$$A(x) = \{l_1(x), \dots, l_N(x)\}.$$

Then $A(\Pi_+)$ is an s-dimensional section of $\mathbf{R}_+^N$ by a plane passing through 0, and $A(\Pi_+)$ is linearly isomorphic to Π_+. Transferring f (which may be regarded as given on Π_+) by means of A to $A(\Pi_+)$ and considering the transferred function on $\mathbf{R}_+^N$, we can apply Lemma 2 to it. This implies everything.

For a given convex bounded polyhedron Γ, consider a covering U_j, $1 \leq j \leq r$, by open convex sets such that each U_j contains only one vertex of Γ and intersects only the faces adjacent to this vertex. Denote this vertex by x_j, and let Π_j be the pyramid with vertex x_j, generated by Γ and containing Γ. Let φ_j, $1 \leq j \leq r$, be an infinitely differentiable partition of unity subordinate to the covering $\{U_j\}$, and let $f_j = f\varphi_j$.

Lemma 4. We have: 1) $\sum f_j = f$; 2) $f_j \in C(\Pi_j)$ and is equal to 0 outside $\Pi_j \cap U_j$; 3)

$$\omega_k(f_j; \tau) \leq A(k, \Gamma)\omega_k(f; \tau), \qquad 1 \leq j \leq r.$$

Now let $p^j(x)$ be a polynomial of degree n, constructed for the function f_j according to Lemma 3. Then from this lemma and Lemma 4 it follows that, for $x \in \Gamma$,

$$|f_j(x) - p^j(x)| \leq A\omega_k(f; \lambda_n(x, \Gamma)/n),$$

and, putting $p(x) = \sum p^j(x)$, we prove the theorem for $n \geq s(k-1)$.

It remains to prove Theorem 1 for $n = k - 1$; but in this case it follows from the multidimensional analogue of Whitney' s theorem, proved in (8).

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Note: Figure translations are in progress. See original paper for figures.