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ASYMPTOTIC FORMULAS FOR THE ERROR OF SOLUTIONS OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS BY FUNCTIONAL NUMERICAL METHODS

MATHEMATICS

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Abstract

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MATHEMATICS

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ASYMPTOTIC FORMULAS FOR THE ERROR OF SOLUTIONS OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS BY FUNCTIONAL NUMERICAL METHODS

(Presented by Academician A. N. Tikhonov on 25 XII 1969)

Suppose that in the closed domain, convex in z, $\Pi(|t-t_0| \leq T,\ |z_i(t)-z_i| \leq R)$ there exists a solution of the Cauchy problem for the system

$$dz/dt = f(t,z), \qquad z(t_0) = z_0, \tag{1}$$

where $z(t)$ is an m-dimensional vector.

If, for the integration of problem (1), a numerical method with integration step h and error on an individual step $e_n = O(h^{s+1})$ is applied, then it is assumed that the round-off error allowed on an individual step is $\Gamma_n = O(h^{s+2})$, and that the prescribed vector function $f(t,z)$ in the domain Π is continuous and has continuous partial derivatives with respect to all arguments up to order $s+1$, inclusive. It is also assumed that the points of the approximate solution belong to the domain Π.

The following notation is adopted: z_n is the approximate solution of problem (1); $t_n = t_0 + nh$; n is an integer; h is the integration step; $f_n = f(t_n, z_n)$; $z(t_n)$ is the exact solution of problem (1), taken at $t = t_n$; $u(t)$ is a matrix function $(m \times m)$ satisfying the equation:

$$\frac{du}{dt} = \frac{\partial f}{\partial z} u, \qquad u(t_0) = E,$$

where E is the identity matrix $(m \times m)$; the matrix $\partial f/\partial z = (\partial f_i(t,z)/\partial z_k)$ $(i,k = 1,\dots,m)$.

Theorem 1. *For the approximate solution of problem (1) by the third-order Runge–Kutta method:*

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$$z_{n+1} = z_n + \frac{h}{6}\left[f\left(t_n + h,\ z_n + 2hf\left(t_n + \frac{h}{2},\ z_n + \frac{h}{2}f_n\right) - hf_n\right) + \right.$$

$$\left. +4f\left(t_n + \frac{h}{2},\ z_n + \frac{h}{2}f_n\right) + f_n\right]$$

the following asymptotic formula is valid as $h \to 0$:

$$z_n = z(t_n) + \frac{h^3}{24}\left[z'''(t_n) - u(t_n)z'''(t_0) + \left(\frac{\partial f}{\partial z}\right)_{t=t_n} z''(t_n) - \right.$$

$$\left. -u(t_n)\left(\frac{\partial f}{\partial z}\right)_{t=t_0} z''(t_0) - \int_{t_0}^{t} u(t_n)u^{-1}(\tau)z^{(\mathrm{IV})}(\tau)\,d\tau\right] + O(h^4).$$

Theorem 2. *If the approximate solution of (1) is computed by the fourth-order Runge–Kutta method of the form*

$$z_{n+1} = z_n + \frac{h}{6}(k_{1n} + 2k_{2n} + 2k_{3n} + k_{4n}),$$

$$k_{1n} = f(t_n, z_n), \qquad k_{2n} = f\left(t_n + \frac{h}{2},\ z_n + \frac{h}{2}k_{1n}\right),$$

$$k_{3n} = f\left(t_n + \frac{h}{2},\ z_n + h\frac{k_{2n}}{2}\right), \qquad k_{4n} = f(t_n + h,\ z_n + hk_{3n}),$$

then the approximate solution will be ($h \to 0$):

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$$
\begin{aligned}
z_n = z(t_n) + h^4 \Bigg\{ & \frac{5}{576}\left[z^{(\mathrm{IV})}(t_n) - u(t_n) z^{(\mathrm{IV})}(t_0)\right] + \\
& + \frac{1}{144}\left[\left(\frac{\partial f}{\partial z}\right)_{t=t_n} z'''(t_n) - u(t_n)\left(\frac{\partial f}{\partial z}\right)_{t=t_0} z'''(t_0)\right] \\
& + \frac{1}{96}\left[\left(\frac{\partial f}{\partial z}\right)^2_{t=t_n} z''(t_n) - u(t_n)\left(\frac{\partial f}{\partial z}\right)^2_{t=t_0} z''(t_0)\right. \\
& \qquad \left. + \left(\frac{\partial f}{\partial z}\right)'_{t=t_n} z''(t_n) - u(t_n)\left(\frac{\partial f}{\partial z}\right)'_{t=t_0} z''(t_0)\right] \\
& - \int_{t_0}^{t_n} u(t_n) u^{-1}(\tau)\left[\frac{z^{(\mathrm{V})}(\tau)}{120} + \frac{1}{96}\left(\frac{\partial f}{\partial z}\right)'' z''(\tau) + \frac{1}{48}\left(\frac{\partial f}{\partial z}\right)' z'''(\tau)\right. \\
& \qquad \left. - \frac{1}{192}\frac{\partial^2 f}{\partial z^2}\left(z''(\tau)\right)^2\right] d\tau \Bigg\} + O(h^5).
\end{aligned}
$$

Theorem 3. In the case of applying the fourth-order Runge–Kutta method of the form

$$
z_{n+1} = z_n + \frac{h}{6}(k_{1n} + 4k_{3n} + k_{4n}),
$$

$$
k_{1n} = f(t_n, z_n), \qquad k_{2n} = f\left(t_n + \frac{h}{4},\, z_n + \frac{h}{4}k_{1n}\right),
$$

$$
k_{3n} = f\left(t_n + \frac{h}{2},\, z_n + \frac{h}{2}k_{2n}\right), \qquad k_{4n} = f(t_n + h,\, z_n + hk_{1n} - 2hk_{2n} + 2hk_{3n})
$$

for the approximate solution the expansion holds

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$$
\begin{aligned}
z_n = z(t_n) + h^4 \Bigg\{ & \frac{5}{576}\left[z^{(\mathrm{IV})}(t_n) - u(t_n) z^{(\mathrm{IV})}(t_0)\right] + \frac{1}{144}\left[\left(\frac{\partial f}{\partial z}\right)_{t=t_n} z'''(t_n)\right. \\
& \left. -u(t_n)\left(\frac{\partial f}{\partial z}\right)_{t=t_0} z'''(t_0)\right] + \frac{1}{192}\left[\left(\frac{\partial f}{\partial z}\right)^2_{t=t_n} z''(t_n)\right. \\
& -u(t_n)\left(\frac{\partial f}{\partial z}\right)^2_{t=t_0} z''(t_0) + \left(\frac{\partial f}{\partial z}\right)'_{t=t_n} z''(t_n) \\
& \left. -u(t_n)\left(\frac{\partial f}{\partial z}\right)'_{t=t_0} z''(t_0)\right] - \int_{t_0}^{t_n} u(t_n) u^{-1}(\tau)\left[\frac{z^{(\mathrm{V})}(\tau)}{120}\right. \\
& \left. + \frac{1}{192}\left(\frac{\partial f}{\partial z}\right)'' z''(\tau) + \frac{1}{96}\left(\frac{\partial f}{\partial z}\right)' z'''(\tau)\right] d\tau \Bigg\} + O(h^5), \qquad h \to 0.
\end{aligned}
$$

Theorem 4. The approximate solution computed by the implicit fourth-order method

$$
z_{n+1} = z_n + \frac{h}{6}\left[f_{n+1} + 4f\left(t_n + \frac{h}{2}, \frac{z_{n+1}+z_n}{2} - \frac{hf_{n+1} - hf_n}{8}\right) + f_n\right], \qquad (2)
$$

can be written in the form

$$
z_n = z(t_n) + h^4\left[\frac{z^{(\mathrm{IV})}(t_n) - u(t_n) z^{(\mathrm{IV})}(t_0)}{576} - \frac{1}{720}\int_{t_0}^{t_n} u(t_n) u^{-1}(\tau) z^{(\mathrm{V})}(\tau)\, d\tau\right] + O(h^5), \qquad h \to 0.
$$

Theorem 5. If, to obtain the approximate solution, a sixth-order method is applied (θ is a parameter, $0.5 - 0.5\sqrt{0.6} \le \theta < 0.5 - 0.5\sqrt{0.2}$)

$$
\begin{aligned}
z_{n+1} = z_n + \frac{h}{60}\Bigg[& \frac{-1 + 10\theta - 10\theta^2}{\theta - \theta^2}(f_{n+1} + f_n) + 32\frac{1 - 5\theta + 5\theta^2}{(1-2\theta)^2} f\left(t_n + \frac{h}{2}, v_n\right) \\
& + \frac{f(t_{n+1} - h\theta, u_n) + f(t_n + h\theta, W_n)}{(\theta - \theta^2)(1-2\theta)^2}\Bigg],
\end{aligned}
$$

$$
v_n = \frac{z_{n+1} + z_n}{2} + \frac{h(1 - 8\theta + 8\theta^2)}{64(\theta - \theta^2)}(f_{n+1} - f_n) - \frac{hf(t_{n+1} - h\theta, y_n) - hf(t_n + h\theta, x_n)}{64(\theta - \theta^2)(1-2\theta)},
$$

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$$
\begin{gathered}
y_n = \theta^2[z_n(3-2\theta) + h(1-\theta)f_n] + (1-\theta)^2[z_{n+1}(1+2\theta) - h\theta f_{n+1}],\\
x_n = (\theta-1)^2[(1+2\theta)z_n + h\theta f_n] + \theta^2[z_{n+1}(3-2\theta) + h(\theta-1)f_{n+1}],\\
u_n = \theta^2(1-2\theta)^2[z_n(7-6\theta) - h(\theta-1)f_n] + (1-2\theta)^2(1-\theta)^2\times\\
\times[z_{n+1}(1+6\theta) - h\theta f_{n+1}] + 16\theta^2(\theta-1)^2[v_n - h(\theta-1/2)f(t_n+h/2, v_n)], \quad (3)\\
w_n = (1-\theta)^2(1-2\theta)^2[z_n(1+6\theta) + h\theta f_n] + \theta^2(1-2\theta)^2\times\\
\times[z_{n+1}(7-6\theta) + h(\theta-1)f_{n+1}] + 16(\theta-1)^2\theta^2\times\\
\times[v_n + h(\theta-1/2)f(t_n+h/2, v_n)],
\end{gathered}
$$

then it can be written as the asymptotic formula, as $h \to 0$,

$$
\begin{aligned}
z_n = z(t_n) + h^6\Big\{ &-\frac{1-\theta+\theta^2}{86400}\,[z^{(\mathrm{VI})}(t_n) - u(t_n)z^{(\mathrm{VI})}(t_0)] -\\
&- \frac{\theta-\theta^2}{14400}\left[5\left(\frac{\partial f}{\partial z}\right)'_{t=t_n} z^{(\mathrm{IV})}(t_n) - 5u(t_n)\left(\frac{\partial f}{\partial z}\right)'_{t=t_0} z^{(\mathrm{IV})}(t_0) + \left(\frac{\partial f}{\partial z}\right)'_{t=t_n} z^{(\mathrm{V})}(t_n) -\right.\\
&\left. - u(t_n)\left(\frac{\partial f}{\partial z}\right)_{t=t_0} z^{(\mathrm{V})}(t_0)\right] + \int_{t_0}^{t_n} u(t_n)u^{-1}(\tau)\Big\{\frac{z^{(\mathrm{VII})}(\tau)}{100800} -\\
&- \frac{\theta-\theta^2}{14400}\left[5\left(\frac{\partial f}{\partial z}\right)'' z^{(\mathrm{IV})}(\tau) + 6\left(\frac{\partial f}{\partial z}\right)' z^{(\mathrm{V})}(\tau)\right]\Big\}\,d\tau\Big\} + O(h^7).
\end{aligned}
$$

All the methods of numerical integration considered in the article may be called functional methods, since they are functional approximations of the Maclaurin and Darboux formulas.

Formulas (2) and (3) can be successfully applied for correction and control in Runge–Kutta methods and directly in the solution of boundary-value problems.

The asymptotic error formulas for functional methods were obtained on the basis of ideas concerning the asymptotics of solutions of problem (1) by Runge–Kutta methods ([1–3]). The essence of the error formulas presented in the article is that, for large $|t_n - t_0|$, the error is determined by integral terms and practically does not depend on the remaining components, just as in Adams-type methods, for large $|t_n - t_0|$, the error of the solutions practically does not depend on errors in computing the beginning of the table ([4]).

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Note: Figure translations are in progress. See original paper for figures.

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