REALIZATION OF CYCLES IN COMPACT SYMMETRIC SPACES BY TOTALLY GEODESIC SUBMANIFOLDS

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Abstract

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REALIZATION OF CYCLES IN COMPACT SYMMETRIC SPACES BY TOTALLY GEODESIC SUBMANIFOLDS

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1. The classical problem of realizing cycles by submanifolds has by now received more or less complete illumination in a large number of works of a purely topological character. A natural feature of all these investigations has been the circumstance that by realization of a cycle one meant its realization by a smooth submanifold of the ambient manifold; at the same time any other properties of these submanifolds, for example metric ones, if the ambient manifold is Riemannian, were ignored as topologically non-invariant.

In the present note the problem of realization is considered from positions that take into account the metric aspect of the question of realization, i.e. the method of embedding a submanifold in the ambient Riemannian manifold $\mathfrak{M}^n$. From the set of all smooth submanifolds we single out totally geodesic submanifolds, as the most "homogeneous" models of the cycles being realized. With this approach the problem of realization splits into two parts: 1) given a compact totally geodesic submanifold V in $\mathfrak{M}^n$; what cycle does it realize in $\mathfrak{M}^n$? 2) given a nontrivial cycle in $\mathfrak{M}^n$; can it be realized by a totally geodesic submanifold of the given type?

In the present note both these problems are solved for the case when the ambient manifold is a symmetric space.

2. Let first $\mathfrak{M}^n = \mathfrak{G}$ be a compact connected Lie group and let $V \subset \mathfrak{G}$ be a totally geodesic submanifold. When does it realize a nontrivial cycle in $H_*(\mathfrak{G}; \mathbf{R})$? We begin with the particular case when V is a subgroup in $\mathfrak{G}$. Then it is easy to see that V realizes a nontrivial cycle if and only if V is not homologous to zero in $\mathfrak{G}$ over the real coefficients $\mathbf{R}$, which completely reduces our problem to the consideration of the well-known homomorphism
$\rho_R^*(V; \mathfrak{G}) : H^*(B_{\mathfrak{G}}; \mathbf{R}) \to H^*(B_V; \mathbf{R})$
and to the selection of all those cases when $\rho_R^*(V; \mathfrak{G})$ is an epimorphism. It is known that this problem is completely solved in terms of Lie algebras; namely, $\rho_R^*(V; \mathfrak{G})$ coincides with the homomorphism $i^* : I_G \to I_H$, where I_G, I_H are

the rings of polynomials on the Cartan subalgebras of the algebras G and H, invariant with respect to the Weyl group.

3. Let now V be an arbitrary compact, simply connected, totally geodesic submanifold in $\mathfrak{G}$. If $\mathfrak{G} = \tilde{I}_0(V)$ is the universal covering group over the identity component of the isometry group $I_0(V)$ of the space V, then it is well known (see [1]) that $\mathfrak{G}$ decomposes into the direct product $\tilde{I}_0(V) = \tilde{I}_0(V_1) \times \cdots \times \tilde{I}_0(V_s)$, where $V = V_1 \times \cdots \times V_s$ is the decomposition of V into irreducible factors. It turns out that if $\mathfrak{G}$ is an arbitrary compact group, then the study of embeddings $V \to \mathfrak{G}$ reduces to the case of the embedding $V \to \tilde{I}_0(V)$; namely, the following assertion holds.

Theorem 1. *Let V be a connected, compact, simply connected, totally geodesic submanifold in a compact Lie group $\mathfrak{G}$, and let*

$\{\mathfrak{G}_\alpha\}$ is the collection of all subgroups of the group $\mathfrak{G}$ containing the submanifold V. Then V decomposes into a direct product

$$V = K \times V_1 \times \cdots \times V_r = K \times V',$$

where K is a compact subgroup of the group $\mathfrak{G}$, and each V_i $(1 \leq i \leq r)$ is an irreducible totally geodesic submanifold in $\mathfrak{G}$ and is not a subgroup in $\mathfrak{G}$. This decomposition has the following property: if

$$A(V) = \bigcap_{(\alpha)} \mathfrak{G}_\alpha,$$

then the subgroup $A(V)$ is compact and semisimple, and the universal covering group $\widetilde{A}(V)$ is isomorphic to the direct product of groups:

$$\widetilde{A}(V) \cong K \times \tilde{I}_0(V_1) \times \cdots \times \tilde{I}_0(V_r).$$

It follows from this theorem that V realizes a nontrivial cycle in $\mathfrak{G}$ if and only if all V_i $(1 \leq i \leq r)$ and K (assuming that K and V_i are distinct from a point) realize nontrivial cycles in $\mathfrak{G}$ over $\mathbf{R}$. Since the case $V = K$ has already been considered above, our original problem reduces to the study, from the homological point of view, of embeddings

$$V \to I_0(V),$$

where V is an irreducible compact symmetric space. It is well known [2] that any symmetric space V can be embedded in $I_0(V)$ as a totally geodesic submanifold (the "Cartan model"). All compact symmetric spaces are divided into two classes: spaces of type I and spaces of type II [1]; therefore we shall present the solution of our first problem in the following form: for each irreducible compact symmetric space V we indicate the cycle which it realizes in the group of isometries $I_0(V)$.

Theorem 2. Let

$$i : V \to I_0(V)$$

be an embedding of a compact irreducible symmetric space V as a totally geodesic submanifold in the maximal group of isometries $I_0(V)$. Then this embedding is the Cartan embedding, and the homological characteristics of these embeddings are listed below:

1) Any space V of type II always realizes a nontrivial cycle in $H_*(I_0(V); \mathbf{R})$.
2) Among spaces V of type I, only the following spaces realize nontrivial cycles in $H_*(I_0(V); \mathbf{R})$:
2a) $V = {}^0A_{2m}\mathrm{I},\ m \geq 1$;
2b) $V = {}^0A_{2m-1}\mathrm{II},\ m \geq 2$;
2c) $V = {}^0D_l\mathrm{II},\ l \geq 4$;
2d) $V = {}^0E_6\mathrm{IV}$.

For all other spaces V of type I, $i([V]) = 0$.

If V is simply connected, then the following more complete assertion holds for spaces of type I realizing nontrivial cycles in $H_*(I_0(V); \mathbf{R})$.

Theorem 3. Let

$$V \to \tilde{I}_0(V)$$

be an embedding of a compact irreducible simply connected symmetric space of type I as a totally geodesic submanifold in $\tilde{I}_0(V)$.

1) $V = \mathrm{SU}(2m+1)/\mathrm{SO}(2m+1)$, $m \geq 1$, (${}^0A_{2m}\mathrm{I}$); let

$$C = x_5 x_9 x_{13} \dots x_{4m+1} \in H^*(\tilde{I}_0(V); \mathbf{Z}_p),$$

where $p \neq 2$ and is prime, if $p \neq 0$. Then:

$$[V] = i^*(N^{-1}C),$$

where $N = 2^m \pmod{p}$.

2) $V = \mathrm{SU}(2m)/\mathrm{Sp}(2m)$, $m \geq 2$, (${}^0A_{2m-1}\mathrm{II}$); let

$$C = x_5 x_9 x_{13} \dots x_{4m-3} \in H^*(\tilde{I}_0(V); \mathbf{Z}).$$

Then

$$i^*(C) = N[V],$$

where $N = 2^{m-1}$.

3) $V = \mathrm{SO}(2l)/\mathrm{SO}(2l-1)$, $l \geq 4$, (${}^0D_l\mathrm{II}$); let

$$C = \bar{x}_{2l-1} \in H^*(\tilde{I}_0(V); \mathbf{Z}_p),$$

where $p \neq 2$ and is prime, if $p \neq 0$. Then:

$$[V] = i^*\left(\frac{1}{2}C\right).$$

4) $V = E_6/F_4(^0E_6\mathrm{IV})$; let

$$C = x_9 - x_{17} \in H^*(\tilde{I}_0(V); \mathbf{Z}_p),$$

where $p \geq 7$ and is prime, if $p \neq 0$. Then

$$[V] = i^*\left(\frac{1}{4}C\right).$$

Remark. As generators $\{x_\alpha\} \in H^*(I_0(V); G)$, canonical primitive generators have been chosen.

Let V be embedded in $\widetilde{A}(V)$, and let $V = K \times V'$ be the decomposition indicated in Theorem 1; let $\mathfrak{H}$ be a stationary subgroup of the manifold V'.

Corollary 1. The manifold V, under the embedding

$$V \to \widetilde{A}(V),$$

realizes a nontrivial cycle in $H_*(\widetilde{A}(V); \mathbf{R})$ if and only if the subgroup $K \times \mathfrak{H}$ is completely nonhomologous to zero in $\widetilde{A}(V)$ for real coefficients.

4. In proving Theorems 2 and 3, a construction related to the Cartan model

$$V \subset I_0(V)$$

is used. Let σ be an involutive auto-

an automorphism of the group $I_0(V)$ determining the space V, i.e., the set of points g, $\sigma(g) = g$, is a subgroup $\mathfrak{H}$, where $V = I_0(V)/\mathfrak{H}$. Then the mapping $p : I_0(V) \to V$, $p(g) = g\sigma(g^{-1})$ [1], determines the principal bundle space $\mathfrak{H} \to I_0(V) \to V$. If $i : V \to I_0(V)$ is the embedding, then $f = pi$ is the "squaring" of the manifold V: $f(v) = v^2$, $v \in V$. It turns out that V realizes a nontrivial cycle in $I_0(V)$ if and only if $\deg f \neq 0$, which reduces the original problem to computing the number $\deg f$. It turns out that this number is closely connected with the geometry of symmetric spaces. The study of the space $^0E_6\mathrm{IV}$ is carried out by means of the properties of the Jordan algebra M_3^8 (see [4]).

Corollary 2. Let V be a simply connected totally geodesic submanifold in an arbitrary compact Lie group $\mathfrak{G}$. Suppose that V realizes a cohomology generator in the ring $H^*(\mathfrak{G}; \mathbb{R})$. Then V is diffeomorphic to one of the following manifolds: S^{2l-1}, $l \geq 2$; $\mathrm{SU}(3)/\mathrm{SO}(3)$, and each of these manifolds V realizes a generator in $H^*(I_0(V); \mathbb{R})$.

The geometric meaning of Corollary 1 is clarified as follows.

Corollary 3. Let $V = V'$ (see Theorem 1) and let $\mathfrak{H} \subset \widetilde{A}(V)$ be a stationary subgroup of the manifold V. If by $\mathrm{ind}(x, y)$ one denotes the intersection index of two cycles x and y of complementary dimension, then

$$\mathrm{ind}(i_*([V]), j_*([\mathfrak{H}'])) = \deg f$$

for some adjacentness class $\mathfrak{H}' = g_0\mathfrak{H}$.

Since every symmetric space $\mathfrak{M}^n$ can be embedded as a Cartan model in the group $\widetilde{I}_0(\mathfrak{M}^n)$, our first problem for an arbitrary symmetric space reduces to the case considered above, $\mathfrak{M}^n = \mathfrak{G}$.

5. Let $\mathfrak{M}^n$ be a compact symmetric space; consider the group $\pi_*(\mathfrak{M}^n) \otimes \mathbb{Q}$, $\pi_*(\mathfrak{M}^n) = \bigoplus_i \pi_i(\mathfrak{M}^n)$, where $\pi_i(\mathfrak{M}^n)$ are the homotopy groups of the space $\mathfrak{M}^n$. Consider the second problem (see above), choosing as the representing submanifold a sphere S^p, i.e., let us decide which nontrivial elements of the group $\pi_*(\mathfrak{M}^n) \otimes \mathbb{Q}$ can be realized by totally geodesic spheres. If we then put $\mathfrak{M}^n = \mathfrak{G}$, we automatically obtain a complete description of the cycles $x \in H^*(\mathfrak{G}, \mathbb{Q})$, $x \neq 0$, realized by totally geodesic spheres.

Denote by $Q_N(\mathfrak{M}^n) = (\pi_*(\mathfrak{M}^n) \otimes \mathbb{Q})_N$ the subgroup in $\pi_*(\mathfrak{M}^n) \otimes \mathbb{Q}$ generated by all elements x, $\dim x \leq N$. Consider an integer n and put $k = k(n) = [1 + \log_2 n]$. Define the function:

$$f_i(n) = f_i(2^{k-1}) = \begin{cases} 2k - i - 1, & \text{if } k \equiv 0 \pmod 4, \\ 2k - i - 3, & \text{if } k \equiv 1 \pmod 4, \\ 2k - i - 5, & \text{if } k \equiv 2 \pmod 4, \\ 2k - i - 3, & \text{if } k \equiv 3 \pmod 4. \end{cases}$$

Theorem 4. Let $\mathfrak{G}$ be one of the following groups: $\mathrm{SU}(n)$, $\mathrm{SO}(n)$, $\mathrm{Sp}(2n)$; $H^*(\mathfrak{G}; \mathbb{Q}) = \Lambda(x_{2k_1-1}, \dots, x_{2k_R-1})$, $R = \operatorname{rank} \mathfrak{G}$. Then the only elements $x \in H^*(\mathfrak{G}; \mathbb{Q})$, $x \neq 0$, realized by totally geodesic spheres in the group $\mathfrak{G}$, are the following elements:

1) if $\mathfrak{G} = \mathrm{SU}(n)$, $n \geq 2$, then $\{x_3, x_5, x_7, \dots, x_{2k(n)-1}\}$;

2) if $\mathfrak{G} = \mathrm{SO}(n)$, $n \geq 8$, then $\{x_{4\alpha-1}\}$, where $3 \leq 4\alpha - 1 \leq f_0(n)$;

3) if $\mathfrak{G} = \mathrm{Sp}(2n)$, $n \geq 1$, then $\{x_{4\alpha-1}\}$, where $3 \leq 4\alpha - 1 \leq f_4(8n)$.

Remark. If $\mathfrak{G} = \mathrm{SO}(n)$, then the formulation of Theorem 4 can be slightly modified: only the elements $\{x_{4\alpha-1}\}$, where $3 \leq 4\alpha - 1 \leq s(2^{k(n)-1})$, are realized by totally geodesic spheres in the group $\mathrm{SO}(n)$; here $s(p)$ denotes the maximum number of linearly independent vector fields on the sphere S^{p-1}. We note that $f_0(n) \leq s(2^{k(n)-1})$.

6. **Theorem 5.** Let V be a compact irreducible symmetric space of type I whose motion group $I_0(V)$ is not an exceptional Lie group. Then the only elements of the group $\pi_*(V) \otimes \mathbb{Q}$, reali-

realized by totally geodesic spheres, are the following elements (here $k = k(n)$):

1) if $V = \mathrm{SU}(2n)/\mathrm{S}(\mathrm{U}(n) \times U(n))$, $k \geqslant 3$,

$$Q_{2k}(V) = \bigoplus_{1 \leqslant \alpha \leqslant k} Q(x_{2\alpha});$$

then $\{x_{2\alpha}\}$, where $1 \leqslant \alpha \leqslant k$.

2) if $V = \mathrm{SO}(2n)/\mathrm{S}(\mathrm{O}(n) \times \mathrm{O}(n))$, $k \geqslant 6$,

$$Q_{2k}(V) = \bigoplus_{1 \leqslant \alpha \leqslant k/2} Q(x_{4\alpha});$$

then $\{x_{4\alpha}\}$, where $4 \leqslant 4\alpha \leqslant f_7(16n)$.

3) if $V = \mathrm{Sp}(2n)/\mathrm{Sp}(n) \times \mathrm{Sp}(n)$, n is even, $k \geqslant 6$,

$$Q_{2k}(V) = \bigoplus_{1 \leqslant \alpha \leqslant k/2} Q(x_{4\alpha});$$

then $\{x_{4\alpha}\}$, where $4 \leqslant 4\alpha \leqslant f_3(4n)$.

4) if $V = \mathrm{SU}(n)/\mathrm{SO}(n)$, $k \geqslant 5$,

$$Q_{2k-1}(V) = \bigoplus_{1 \leqslant \alpha \leqslant (k-1)/2} Q(x_{4\alpha+1});$$

then $\{x_{4\alpha+1}\}$, where $5 \leqslant 4\alpha + 1 \leqslant f_6(8n)$.

5) if $V = \mathrm{SU}(2n)/\mathrm{Sp}(2n)$, $k \geqslant 5$,

$$Q_{2k+1}(V) = \bigoplus_{1 \leqslant \alpha \leqslant k/2} Q(x_{4\alpha+1});$$

then $\{x_{4\alpha+1}\}$, where $5 \leqslant 4\alpha + 1 \leqslant f_2(4n)$.

6) if $V = \mathrm{SO}(2n)/U(n)$, $k \geqslant 5$,

$$Q_{2k}(V) = \bigoplus_{1 \leqslant \alpha \leqslant (k-1)/2} Q(x_{4\alpha+2});$$

then $\{x_{4\alpha+2}\}$, where $2 \leqslant 4\alpha + 2 \leqslant f_1(2n)$.

7) if $V = \mathrm{Sp}(2n)/U(n)$, $k \geqslant 5$,

$$Q_{2k}(V) = \bigoplus_{1 \leqslant \alpha \leqslant (k-1)/2} Q(x_{4\alpha+2});$$

then $\{x_{4\alpha+2}\}$, where $2 \leqslant 4\alpha + 2 \leqslant f_5(8n)$.

Remark. Theorem 5 presents all series of compact symmetric spaces V for which $I_c(V)$ is not an exceptional Lie group. However, the technique by means of which Theorem 5 was obtained makes it easy to resolve the question also in the case of exceptional groups. The realization theorem for the manifolds $\mathfrak{G}_{n,k}(F)$, where $F = \mathbf{R}, \mathbf{C}, \mathbf{H}$, can be obtained from items 1), 2), 3) of Theorem 5, using the mutual inclusions of Grassmann manifolds.

The proof of Theorems 4, 5 uses the results set forth in ([3]), and is divided into two parts: the construction of realizations and the proof of the uniqueness (up to a factor $q \neq 0$, $q \in \mathbf{Q}$) of those elements which admit this realization. The results of ([3]) are supplemented by the construction of one more totally geodesic sphere S^{2k-1}, realizing a nontrivial cycle in $H_*(\mathrm{SO}(n); \mathbf{Q})$, where $k = [1+\log_2 n]$, $k \geqslant 4$, $k \equiv 0 (\mathrm{mod}\, 4)$.

This construction, as well as the proofs of the second point, make essential use of the theory of spinor representations of the orthogonal group and Theorem 1. We note that the absence of cycles realized by totally geodesic spheres in dimensions greater than the bounds indicated in Theorems 4, 5 ($\sim 2\log_2 n$) is connected with the fact that all symmetric spaces in general contain no totally geodesic spheres, beginning approximately with these dimensions.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.