MULTIPLIER TRANSFORMATIONS FOR PSEUDODIFFERENTIAL OPERATORS IN \(L_p\)

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Abstract

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MULTIPLIER TRANSFORMATIONS FOR PSEUDODIFFERENTIAL OPERATORS IN L_p

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1. By a multiplier transformation we shall mean an operation that assigns to every pseudodifferential (ps.d.) operator K with symbol $K(\xi, x)$, i.e.

$$(Ku)(x) = (2\pi)^{-m/2} \int_{R^m} K(\xi, x) e^{i\langle\xi,x\rangle} \hat{u}(\xi)\, d\xi, \tag{1}$$

a new ps.d. operator ΦK with symbol $\varphi(\xi, x)K(\xi, x)$, i.e.

$$(\Phi K u)(x) = (2\pi)^{-m/2} \int_{R^m} \varphi(\xi, x) K(\xi, x) e^{i\langle\xi,x\rangle} \hat{u}(\xi)\, d\xi. \tag{2}$$

Here R^m is m-dimensional Euclidean space; x and ξ are points of R^m; $\langle\xi, x\rangle = \xi_1 x_1 + \dots + \xi_{mx} m$; $\hat{u}$ is the Fourier transform of the function $u \in S$, where S is the space of rapidly decreasing functions. For symbols of ps.d. operators homogeneous of degree 0 (with respect to ξ), an operator of the form (1) is a singular integral (s.i.) operator.

The question is posed: under what conditions does the transformation $K \to \Phi K$ preserve the continuity of ps.d. operators in various function spaces? For the spaces $L_2(R^m)$ and $W_2^s(R^m)$ of Sobolev-Slobodetskii this problem was investigated in papers ([1–5]). The purpose of the present note is to report two results on multiplier transformations in the spaces $L_p(R^m)$.

2. We shall give a description of the function spaces in terms of which the conditions of Theorems 1 and 2 will be expressed.

a) Introduce the rings $\mathfrak{A}_l = \{\xi \in R^m : 2^{l-2} \le |\xi| \le 2^{l+2}\}$, $l = 0, \pm 1, \dots$. By $L_p^{\alpha,\beta}$ $(\alpha, \beta \ge 0)$ we denote the space of all generalized functions $\varphi(\xi)$ belonging to $L_p^{\alpha}(\mathfrak{A}_l)$ (see ([1])) for every l, and for which the seminorm $\|\varphi\|_{L_p^{\alpha,\beta}}$ is finite, where

$$\|\varphi\|^p_{L_p^{\alpha,\beta}} = \sum_{l=-\infty}^{\infty} 2^{l\beta p} \|\varphi\|^p_{L_p^\alpha(\mathfrak{A}_l)}.$$

If $\alpha = N$ is an integer, then $\|\varphi\|_{L_p^{N,\beta}}$ is equivalent to the integral seminorm

$$\left(\sum_{|\nu|=N} \int_{R^m} |\xi|^{|\beta p|} \, |D^\nu \varphi(\xi)|^p \, d\xi \right)^{1/p}.$$

b) By $L_{\infty;x}(L_p^{\alpha,\beta};\xi)$ we shall denote the space of all functions $\varphi(\xi,x)$ belonging for almost all $x \in R^m$ to the space $L_p^{\alpha,\beta}$ (with respect to—

-tively variable ξ) and for which the seminorm is finite

$$\|\varphi\|_{L_{\infty;x}(L^{\alpha,\beta}_{p;\xi})} \equiv \operatorname{vrai\,sup}_{x \in R^m} \|\varphi(\cdot,x)\|_{L_p^{\alpha,\beta}}.$$

c) Let S^{m-1} be the unit sphere of the space R^m; θ its points. By $W_p^\alpha(S^{m-1})$ we denote the Sobolev–Slobodetskii space of functions $\varphi(\theta)$ defined on the sphere S^{m-1}. The space $L_{\infty;x}(W^\alpha_{p;\theta}(S^{m-1}))$ is defined by analogy with item b).

3. We formulate the results of the note.

Theorem 1. *Suppose that the following two conditions are satisfied:*

$$\text{(I)} \quad \varphi(\xi,x) \in L_{\infty;x}(W^\alpha_{2;\xi}(R^m)) \quad \text{for some } \alpha > m/2.$$

$$\text{(II)} \quad \varphi(\xi,x) \in L_{\infty;x}(L^{m/2+\beta,\beta}_{2;\xi}) \quad \text{for some } \beta > 0.$$

If $|1/p - 1/2| < \beta/m$ $(1 < p < \infty)$, then the following conclusion is valid:

For every operator K of the form (1) with locally summable symbol $K(\xi,x)$, continuous in $L_p(R^m)$, the operator ΦK of the form (2) is also continuous in $L_p(R^m)$. Moreover

$$\|\Phi K\|_{L_p \to L_p} \le C \left(\|\varphi\|_{L_\infty(W_2^\alpha)} + \|\varphi\|_{L_\infty(L_2^{m/2+\beta,\beta})} \right) \|K\|_{L_p \to L_p}.$$

Theorem 2. *Let the function $\varphi(\xi,x)$ be homogeneous of degree 0 in ξ. If* *

$$\varphi(\theta,x) \in L_{\infty;x}(W^\alpha_{2;\theta}(S^{m-1})), \quad \text{then, for } \alpha > (m-1)/2 + m|1/p - 1/2|$$

$$(1 < p < \infty)$$

the conclusion of Theorem 1 is valid. Moreover

$$\|\Phi K\|_{L_p\to L_p} \le C\|\varphi\|_{L_\infty(W_2^\alpha)}\|K\|_{L_p\to L_p}.$$

4. In the space $L_2(R^m)$ the multiplier problem formulated above is included in the theory of double operator integrals ([1], [4]), the use of which leads to various results. In studying the same problem in the spaces $L_p(R^m)$, the general methods of Hilbert space applied in ([1–4]) are unsuitable. However, some elements of the method of integral sums developed in ([1]) can be used in proving Theorems 1 and 2. The idea consists in approximating the multipliers $\varphi(\xi,x)$ by functions of a special form—piecewise-polynomial approximations corresponding to suitable decompositions of the space R^m. This idea is combined with the application of new results of Petre ([6]) and Littman, McCarthy, and Rivière ([7]) on multipliers of Fourier integrals and on estimates of Littlewood–Paley type.

5. Since the symbol of the identity operator is equal to $K(\xi,x)\equiv 1$, our results contain the following assertion.

Corollary. *Under the conditions of Theorems 1 and 2, a ps.d. (s.i.) operator with symbol $\varphi(\xi,x)$ is bounded in $L_p(R^m)$.*

We note that the boundedness criterion for a s.i. operator in $L_p(R^m)$ that follows as a special result from Theorem 2, for $p<2$, somewhat improves the result of S. G. Mikhlin (([8]), Theorem V.1.26). For $p>2$ the conditions of Theorem 2 are more restrictive than the conditions of S. G. Mikhlin; this is connected with the fact that the latter are not of multiplier character.

Other boundedness conditions for ps.d. operators in $L_p(R^m)$ were indicated, in the case $p=2$, by L. Hörmander ([9]); from this V. M. Kagan ([10]) derived an analogous criterion for arbitrary p. Both results likewise do not have multiplier character. Kagan' s conditions and Theorem 1 of the present work do not cover one another.

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[*] The restriction of the function $\varphi(\xi,x)$ to the sphere S^{m-1} (with respect to the first variable) will be denoted by $\varphi(\theta,x)$.

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Note: Figure translations are in progress. See original paper for figures.