

# ON THE THEORY OF PERIODIC SOLUTIONS OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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**Abstract**

**Full Text**

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*MATHEMATICS*

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## ON THE THEORY OF PERIODIC SOLUTIONS OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

*(Presented by Academician I. G. Petrovsky on 31 XII 1969)*

Consider the system of ordinary differential equations

$$dx/dt = F(t, x), \quad (1)$$

where  $x = \{x_1, \dots, x_n\}$  is a point of the  $n$ -dimensional space  $R^n$ , and the vector-function  $F(t, x)$  is defined and continuous jointly in the variables  $t, x$  in the domain  $(-\infty < t < \infty, x \in R^n)$  and is  $\omega$ -periodic in  $t$ . As is known (see <sup>(1)</sup>), the question of the existence of  $\omega$ -periodic solutions of system (1) can be reduced to the question of the existence of fixed points of certain integral operators acting in various function spaces. An example of such operators is the completely continuous operator  $A$ , defined by the equality

$$Ax(t) = x(\omega) + \int_0^t F[sx(s)] ds, \quad (2)$$

acting in the space  $C[0, \omega]$  of vector-functions continuous on the interval  $[0, \omega]$ . It is not difficult to see that the fixed points of the operator (2) (and only they) determine the  $\omega$ -periodic solutions of system (1). Thus, the question of the existence of  $\omega$ -periodic solutions of system (1) is equivalent to the question of the existence of zeros of the completely continuous vector field

$$\Phi x(t) = x(t) - Ax(t). \quad (3)$$

Conditions for the existence of zeros of completely continuous vector fields are given by the following general topological principle (see <sup>(2,3)</sup>).

*Let the completely continuous vector field  $\Phi$ , defined on the set  $\bar{\Omega} = \Omega \cup \Gamma$ , where  $\Omega$  is some bounded domain in the Banach space  $C[0, \omega]$  with boundary  $\Gamma$ , have no zero vectors on  $\Gamma$ . Let the rotation  $\gamma[\Phi; \Gamma]$  of the field  $\Phi$  on the boundary  $\Gamma$  be different from zero. Then there exists a point  $x_0 \in \Omega$  such that  $\Phi x_0 = \theta$ .*

The present paper is devoted to the calculation of the rotation of the completely continuous vector field (3) for certain classes of systems of ordinary differential equations.

1. Let system (1) have the form

$$dx/dt = P(t, x) + f(t, x), \quad (4)$$

where the vector-function  $P(t, x)$  is continuous jointly in the variables  $t, x$ ,  $\omega$ -periodic in  $t$ , and positively homogeneous in  $x$  of order  $m > 0$  ( $P(t, \lambda x) = \lambda^m P(t, x)$  for  $\lambda \geq 0$ ), while the continuous vector-function  $f(t, x)$  satisfies the condition

$$\lim_{\|x\| \rightarrow \infty} \left\{ \|x\|^{-m} \sup_{0 \leq t \leq \omega} \|f(t, x)\| \right\} = 0.$$

Alongside system (4), consider the system

$$dx/dt = P(t, x). \quad (5)$$

We shall call system (5) an  $m$ -system if:

a) for  $m > 1$  the autonomous system

$$dy/dt = P(t_0, y)$$

for every  $t_0 \in [0, \omega]$  has no nonzero solutions bounded on the entire axis  $(-\infty, +\infty)$ ;

b) for  $m = 1$  system (5) has no nonzero  $\omega$ -periodic solutions;

c) for  $m < 1$  the autonomous system

$$\frac{dy}{dt} = \int_0^\omega P(s, y) ds$$

has no nonzero stationary solutions.

As was already noted above, the  $\omega$ -periodic solutions of system (4) are determined by the zeros of the completely continuous vector field

$$\Phi_1 x(t) = x(t) - x(\omega) - \int_0^t \{P[s, x(s)] + f[s, x(s)]\} ds. \quad (6)$$

We shall say that a vector field  $\Phi$  is nondegenerate at infinity if  $\Phi x \neq 0$  outside some ball.

Let a completely continuous vector field  $\Phi$  be nondegenerate at infinity. Then the rotations  $\gamma[\Phi; S_\rho]$  of the field  $\Phi$  on spheres  $S_\rho$  of sufficiently large radius coincide. This common rotation  $\gamma[\Phi; \infty]$  will be called the rotation of the completely continuous field  $\Phi$  at infinity.

Consider the completely continuous vector field

$$\Phi_0 x(t) = x(t) - x(\omega) - \int_0^t P[s, x(s)] ds. \quad (7)$$

**Theorem 1.** *Let system (5) be an  $m$ -system. Then the vector fields (6) and (7) are nondegenerate at infinity and their rotations coincide:  $\gamma[\Phi_1; \infty] = \gamma[\Phi_0; \infty]$ .*

**2.** Theorem 1 makes it possible to reduce the computation of the rotation of the completely continuous vector field (6) to the computation of the rotation of the simpler vector field (7). In this section we indicate formulas for computing the rotation of the completely continuous vector field (7).

Consider the finite-dimensional vector field

$$\Psi x = - \int_0^\omega P(s, x) ds \quad (x \in R^n). \quad (8)$$

Let the field  $\Psi$  on the unit sphere  $S \subset R^n$  have no zero vectors. Then the rotation  $\gamma[\Psi; S]$  of the field  $\Psi$  on the sphere  $S$  is defined.

**Theorem 2.** *Let the degree of homogeneity  $m$  of the function  $P(t, x)$  be less than one:  $m < 1$ . Let system (5) be an  $m$ -system and let the vector field (8) on the unit sphere  $S$  have no zero vectors.*

*Then the rotation  $\gamma[\Phi_0; \infty]$  of the field (7) at infinity coincides with the rotation  $\gamma[\Psi; S]$  of the field (8) on the unit sphere.*

Suppose that system (5) has no nonzero  $\omega$ -periodic solutions. Then, obviously, the rotation  $\gamma[\Phi_0; \infty]$  of the field (7) at infinity coincides with the index  $\gamma[\Phi_0; \theta]$  of the zero singular point of this field, and the index  $\gamma[\Phi_0; \theta]$  of the zero singular point of the field (7), in the case when  $m > 1$ , is equal to the rotation  $\gamma[\Psi; S]$  of the field (8) on the unit sphere  $S$ . Therefore, the following is valid:

**Theorem 3.** Let  $m > 1$ , and suppose that system (5) has no nonzero  $\omega$ -periodic solutions. Suppose that the field (8) on the unit sphere has no zero vectors.

Then the equality

$$\gamma[\Phi_0; \infty] = \gamma[\Psi; S]$$

holds.

Consider the family of systems of ordinary differential equations

$$dx/dt = P(t, x) + \mu[P(t_0, x) - P(t, x)] \quad (0 \leq \mu \leq 1) \quad (9)$$

into the finite-dimensional vector field  $\Psi_0$ , defined by the equality

$$\Psi_0 x = -P(0, x). \quad (10)$$

**Theorem 4.** Let  $m > 1$ , and suppose there exists  $t_0 \in [0, \omega]$  such that every system of the family (9) is an  $m$ -system.

Then the equality

$$\gamma[\Phi_0; \infty] = \gamma[\Psi_0; S],$$

holds, where  $\gamma[\Psi_0; S]$  is the rotation of the vector field (10) on the unit sphere  $S$ .

Suppose that the function  $P(t, x)$  does not depend on  $t$ , i.e.  $P(t, x) \equiv P(0, x)$ . Then the following is valid.

**Theorem 5.** Let  $m \geq 1$ , and suppose system (5) has no cycles and no nonzero equilibrium states.

Then the rotation  $\gamma[\Phi_0; \infty]$  of the field (7) at infinity coincides with the rotation  $\gamma[\Psi_0; S]$  of the field (10) on the unit sphere  $S$ .

We note that, for computing or estimating the rotation of finite-dimensional vector fields, one can use the results of the works <sup>(4,5)</sup>.

3. We now formulate one general criterion for the existence of periodic solutions of system (4).

**Theorem 6.** Let the function  $P(t, x)$  be such that system (5) is an  $m$ -system. Suppose that the rotation  $\gamma[\Phi_0; \infty]$  of the field (7) at infinity is different from zero.

Then system (4) has at least one  $\omega$ -periodic solution.

The proof of this theorem follows directly from the general topological principle given above and the assertion of Theorem 1.

We give some criteria for the rotation of the field (7) to be different from zero (see <sup>(3,6)</sup>).

- 1) Let system (5) be an  $m$ -system. Suppose the vector-function  $P(t, x)$  is odd in  $x$ :

$$P(t, -x) = -P(t, x).$$

Then the rotation  $\gamma[\Phi_0, \infty]$  of the field (7) is odd.

- 2) Let system (5) be an  $m$ -system. Suppose there exists a periodic matrix  $U$  of period  $p$ :  $U^i \neq I$  ( $i = 1, \dots, p-1$ ) and  $U^p = I$ , such that, for the matrices  $U^i$  ( $i = 1, \dots, p-1$ ), unity is not an eigenvalue. Suppose  $P(t, x)$  satisfies the condition

$$P(t, Ux) = UP(t, x) \quad (0 \leq t \leq \omega, x \in R^n).$$

Then  $\gamma[\Phi_0; \infty] = 1 + kp$ , where  $k$  is some integer.

From assertions 1), 2), and Theorem 6 it follows:

**Theorem 7.** Let the vector-function  $P(t, x)$  be such that system (5) is an  $m$ -system. Suppose condition 1) or 2) is fulfilled.

Then system (4) has at least one  $\omega$ -periodic solution.

In conclusion, we note that Theorem 1 can be used according to the standard scheme (see (1)) to prove the existence of second periodic solutions of system (4).

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