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Abstract

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ON THE DICHOTOMY OF SOLUTIONS OF FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH ALMOST PERIODIC COEFFICIENTS

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In papers ([1–3]) the Green' s function was constructed for the problem on almost periodic solutions of ordinary differential equations. The question of constructing the Green' s function is closely connected with the so-called exponential dichotomy of solutions of the homogeneous equation. Determining the conditions under which an exponential dichotomy of solutions exists for various concrete classes of equations with distributed parameters is a very difficult problem. The main difficulty here consists in the preliminary splitting of the space of initial conditions into the direct sum of two subspaces corresponding to bounded and unbounded, as $t \to \infty$, solutions of the homogeneous equation. In a number of works ([1,4]) devoted to the problem of dichotomy on a half-axis and on the whole axis for equations in a Banach space with bounded and unbounded operators, such a splitting was assumed a priori to have been carried out.

The authors have found a device that makes it possible to establish the dichotomy of solutions for certain classes of equations with distributed parameters. Using it, the Green' s function is constructed for the problem on almost periodic solutions of functional-differential equations. Here an essential use is made of the passage to equations in Banach spaces that are equivalent in a certain sense ([5]).

As in the case of ordinary differential equations ([6–8]), the results obtained make it possible to solve a number of questions in the local and nonlocal theory of almost periodic oscillations for functional-differential equations.

1. Let C denote the Banach space of continuous and bounded functions on the entire axis with values in the n-dimensional space R^n, and let B denote its subspace consisting of almost periodic functions. By B_1 we shall denote the totality of almost periodic functions possessing an almost periodic derivative.

For each $t \in (-\infty, \infty)$, consider the linear operator

$$l(t, x(s)) = \int_{-h}^{0} [d_s r(t, s)] x(s),$$

acting from the space $C(-h, 0)$ of functions continuous on the interval $[-h, 0]$ into R^n. Here $r(t, s)$ is a square matrix of order n, whose elements, for each t, are functions of bounded variation, $r(t, -h) \equiv 0$.

Below we shall assume that the following conditions are satisfied:

1. The matrix $r(t, 0)$ is almost periodic;
2. $\sup \left| \bigvee_{-h}^{0} r(t, s) \right| < \infty, \quad -\infty < t < \infty$, where $| \ |$ is some norm of a matrix in R^n;
3. The elements of the matrix $r(t, s)$ are almost periodic in t as abstract functions with values in the space of summable functions $L(-h, 0)$.

Let us now consider in the space C the linear operator

$$Lx(t) = x'(t) + l(t, x(t + s)), \tag{1}$$

which we define on those functions $x(t) \in C$ for which $Lx(t) \in B$.

The meaning of conditions 1–3 is clarified in the following lemmas, in whose proofs some constructions from (9) are used.

Lemma 1. Let $x(s)$ be an arbitrary function from $C(-h, 0)$. Then $l(t, x(s)) \in B$ if and only if conditions 1–3 are satisfied.

Lemma 2. The operator $\Pi x(t) = l(t, x(t + s))$ acts and is continuous in B if and only if conditions 1–3 are satisfied.

Lemma 3. The domain of definition of the operators L contains B_1 if and only if conditions 1–3 are satisfied.

We shall call the operator L regular if the equation

$$Lx = f(t) \tag{2}$$

has, for every function $f(t) \in B$, at least one solution $x(t) \in C$.

Theorem 1. If the operator L is regular, then equation (2), for $f(t) \in C$ ($f(t) \in B$), has a unique solution $x(t) \in C$ ($x(t) \in B$).

From Theorem 1 it follows, in particular, that the domain of definition of the operator L coincides with B_1.

2. Let $x(t)$ be a solution of the homogeneous equation

$$Lx = 0 \tag{3}$$

with initial function $x(s) \in C(-h, 0)$, which is prescribed at the time τ. Define in $C(-h, 0)$ the family of operators

$$U(t, \tau)x(s) = x(t + s). \tag{4}$$

We shall say that for equation (3) there is an exponential dichotomy of solutions if the following conditions are satisfied.

First, in the space $C(-h, 0)$ there exist projectors $P_+(t)$ and $P_-(t)$, depending almost periodically on t in the uniform operator topology, such that

$$P_+(t)P_-(t) = 0, \qquad P_+(t) + P_-(t) = I; \tag{5}$$

$$P_+(t)U(t, \tau) = U(t, \tau)P_+(\tau), \qquad P_-(t)U(t, \tau) = U(t, \tau)P_-(\tau), \tag{6}$$

where $P_-(t)$ is a projector onto a finite-dimensional subspace.

Second, there exist positive constants M_1 and γ_1 such that

$$\|U(t, \tau)P_+(\tau)x(s)\| \leq M_1 \exp[-\gamma_1(t - \tau)]\|P_+(\tau)x(s)\| \qquad (t \geq \tau). \tag{7}$$

Third, the solutions of equation (3) with initial conditions $P_-(\tau)x(s)$ are defined both for $t \geq \tau$ and for $t \leq \tau$, and the inequalities

$$\|U(t, \tau)P_-(\tau)x(s)\| \geq M_2 \exp[+\gamma_2(t - \tau)]\|P_-(\tau)x(s)\| \qquad (t \geq \tau), \tag{8}$$

$$\|U(t, \tau)P_-(\tau)x(s)\| \leq M_3 \exp \gamma_3(t - \tau)\|P_-(\tau)x(s)\| \qquad (t \leq \tau). \tag{9}$$

hold, where $M_2, M_3, \gamma_2, \gamma_3$ are some positive constants.

In inequalities (7)-(9), $\| \ \|$ is the norm in $C(-h, 0)$.

Theorem 2. If the operator (1) is regular, then there is an exponential dichotomy of solutions of equation (3).

3. Denote by $C_{\text{pc}}(-h, 0)$ the space of bounded piecewise-continuous on $[-h, 0]$ functions with values in R^n. We shall assume that in this space $\|x\| = \sup |x(s)|$, $(-h \leq s \leq 0)$, where $| \ |$ is the norm in R^n.

It is clear that the operator (4) is defined on functions $x(s) \in C_{\text{pc}}(-h, 0)$. Define the projectors $P_+(t)$ and $P_-(t)$ on functions $x(s) \in C_{\text{pc}}(-h, 0)$ as follows:

$$P_-(t)x(s) = U(t, h+t)P_-(h+t)U(h+t, t)x(s),$$

$$P_+(t)x(s) = x(s) - P_-(t)x(s).$$

Obviously, formulas (5)–(6) and inequalities (7)–(9) remain valid.

Denote by $e_i(s)$, $(i = 1, \dots, n)$, the elements of the space $C_{\text{kn}}(-h, 0)$ for which the i-th component is equal to zero for $-h \leq s < 0$ and is equal to one for $s = 0$, while the remaining components are identically equal to zero. Next, consider in $C_{\text{kn}}(-h, 0)$ the family of operators

$$G(t, \tau) = \begin{cases} U(t, \tau)P_+(\tau), & \text{for } t \geq \tau, \\ -U(t, \tau)P_-(\tau), & \text{for } t < \tau, \end{cases}$$

and denote by $g(t, \tau)$ the matrix-function whose i-th column-vector is determined by the formula $l_0(G(t, \tau)e_i(s))$, where $l_0(x(s)) = x(0)$ is the operator from $C_{\text{kn}}(-h, 0)$ into R^n.

According to Theorem 1, the operator L is continuously invertible in B.

Theorem 3. The formula

$$L^{-1}f(t) = \int_{-\infty}^{\infty} g(t, \tau)f(\tau)\, d\tau$$

is valid.

4. We now consider the family of operators

$$L(\mu)x(t) = x'(t) + l(t, x(t+s); \mu),$$

where

$$l(t, x(s); \mu) = \int_{-h}^{0} [d_s r(t, s; \mu)]x(s),$$

and the parameter μ belongs to some metric compact set M. We shall assume that for each $\mu \in M$, $r(t, s; \mu)$ satisfies conditions 1–3.

Theorem 4. Let the following conditions be fulfilled:

1. For each $T > 0$

$$\lim_{\mu\to\mu_0} \sup_{|t-\tau|\le T} \left| \int_\tau^t [r(\sigma,0;\mu) - r(\sigma,0;\mu_0)]\, d\sigma \right| = 0,$$

$$\lim_{\mu\to\mu_0} \sup_{|t-\tau|\le T} \int_{-h}^0 \left| \int_\tau^t [r(\sigma,s;\mu) - r(\sigma,s;\mu_0)]\, d\sigma \right| ds = 0;$$

2.

$$\sup \left[\int_{-h}^0 \dot{V} r(t,s;\mu),\ |r(t,0;\mu)| \right] < \infty, \quad t \in (-\infty,\infty),\ \mu \in M.$$

Let the operator $L(\mu_0)$ be regular.

Then the operators $L(\mu)$ are regular for all values of μ sufficiently close to μ_0; the projectors $P_+(t;\mu)$ and $P_-(t;\mu)$ converge uniformly with respect to t, respectively, to the projectors $P_+(t;\mu_0)$ and $P_-(t;\mu_0)$; for some $\gamma_0 > 0$ the equality

$$\lim_{\mu\to\mu_0} \sup_{-\infty<t,\tau<\infty} |g(t,\tau;\mu) - g(t,\tau;\mu_0)| \exp[-\gamma_0 |t-\tau|] = 0$$

holds.

In verifying condition 1, the following may be useful.

Lemma 4. Condition 1 is fulfilled if and only if, for every $T > 0$ and for any function $x(s) \in C(-h,0)$,

$$\lim_{\mu\to\mu_0} \sup_{|t-\tau|\le T} \left| \int_\tau^t [l(\sigma,x(s);\mu) - l(\sigma,x(s);\mu_0)]\, d\sigma \right| = 0.$$

5. Finally, we consider the higher-order operator

$$Lx(t) = x^{(m)}(t) + l_1(t, x^{(m-1)}(t+s)) + \cdots + l_m(t, x(t+s)), \tag{10}$$

which is defined on those m-times continuously differentiable functions $x(t) \in C$ for which $Lx(t) \in B$. It is assumed that $l_i(t, x(s))$, $(i = 1, \ldots$

$\cdots$, $m)$ satisfy the restrictions indicated in item 1. The regularity of the operator (10) is defined in the same way as the regularity of the operator (1). Along with the operator (10) we consider the corresponding first-order operator

$$\tilde{L}u(t) = u'(t) + \tilde{l}(t, u(t+s)), \tag{11}$$

where $u(t) = (u_1(t), \ldots, u_m(t))$, $(u_i(t) \in C,\ i = 1, \ldots, m)$,

$$\tilde{l}(t, u(s)) = \text{colon}(-u_2(0), \dots, -u_m(0), l_m(t, u_1(s)) + \dots + l_1(t, u_m(s))).$$

Theorem 5. *The operator (10) is regular if and only if the operator (11) is regular.*

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Note: Figure translations are in progress. See original paper for figures.