BIFURCATION POINTS OF THE HAMMERSTEIN EQUATION

MATHEMATICS

1970

SovietRxiv

Abstract

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UDC 517.948.33

MATHEMATICS

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BIFURCATION POINTS OF THE HAMMERSTEIN EQUATION

(Presented by Academician I. N. Vekua on March 10, 1970)

The paper considers the problem of bifurcation points of the equation

$$\int_{\Omega} k(t,s) f[s, x(s)]\, ds = \lambda x(t). \tag{1}$$

Here Ω is a bounded closed subset of a finite-dimensional space. In what follows it is assumed that the function $f(s,u)$, defined on $\Omega \times R^n$ with values in R^n, satisfies the Carathéodory conditions and is potential ($f(s,u) = \operatorname{grad} G(s,u)$; $G(s,0) = 0$); the matrix $k(t,s)$ $(t, s \in \Omega)$ is symmetric, measurable in the aggregate of the variables, and the linear operator defined by it

$$Kx(t) = \int_{\Omega} k(t,s) x(s)\, ds \tag{2}$$

in the Hilbert space of vector functions $H = \mathcal{L}_2$ is self-adjoint and has no more than a finite number of negative eigenvalues (each of finite multiplicity). The paper is a natural continuation of work ([1]) on eigenvectors of the Hammerstein operator.

Equation (1) has the form $Ax = \lambda x$. Suppose that the Hammerstein operator A acts in some real Banach space E. If the condition $f(s,0) = 0$ is fulfilled, then equation (1) has the zero solution for all real λ. A number λ_0 is called a **bifurcation point** of equation (1) if for any $\varepsilon, \delta > 0$ there exists a λ, $|\lambda - \lambda_0| < \varepsilon$, to which there corresponds a nonzero solution $x(t)$ of equation (1), with $\|x\|_E < \delta$. The paper also considers asymptotic bifurcation points (in this case fulfillment of the condition $f(s,0) = 0$ is not assumed). A number λ_0 is called an **asymptotic bifurcation point** of equation (1) if for any $\varepsilon, \rho > 0$ there exists a λ, $|\lambda - \lambda_0| < \varepsilon$, to which there corresponds a solution $x(t)$ of equation (1), with $\|x\|_E > \rho$.

The problem of bifurcation points of equation (1), in the case where the role of E is played by various $\mathcal{L}_p$, was considered in a number of works by M. A.

Krasnosel' skii (^{2,3}) and others, M. M. Vainberg (^{4}) and others. A detailed analysis of the properties of operators in general Banach spaces of measurable functions (the so-called ideal spaces (^{5,6})) made it possible to obtain essentially new results for this problem; these results are presented in the present note.

1. Let E be an ideal space of measurable vector functions finite almost everywhere on Ω (see (^{1,5})). By E' we denote the ideal space dual to E; by E^0, the set of vector functions from E with absolutely continuous norm; by E'/E, the space of multipliers from E into E'. The spaces E^0 and E'/E are themselves ideal. Examples of ideal spaces are the space M_{u_0} (u_0 a nonnegative measurable function) of vector functions for which the norm $\|x\|_{M_{u_0}} = \inf\{\lambda : \ |x| \leq \lambda u_0\}$ is meaningful, the spaces $\mathcal{L}_p$, and Orlicz spaces.

A set in E is called w-bounded if for every $\varepsilon > 0$ this set has an ε-net T such that $|x| \leq u_0$ for all $x \in T$ and for some $u_0 \in E$. An operator acting from one ideal space E_1 into another E_2 is called w-bounded if it maps every norm-bounded set into a w-bounded set.

Let H_1 be the linear span of the eigenvectors of the operator (2) in H corresponding to negative eigenvalues. Denote $J = -P_1 + P_2$, where P_1 and P_2 are the projection operators, respectively, onto H_1 and $H_2 = H \ominus H_1$.

We shall state the conditions that will be assumed to hold in what follows. Let the superposition operator $fx(s) = f[s, x(s)]$ act from E into E', and let the linear operator K (and consequently also $\widetilde{K} = JK$) act from E' into E, with $E \subseteq H$. In this case in (1) the operator $C = \widetilde{K}^{1/2}$ acts from H into E, while the adjoint operator C' acts from E' into H. The operator $G[s, x(s)]$ acts from E into $\mathcal{L}_1$, and therefore on H the Golomb functional is defined by

$$\Phi(y) = \int_{\Omega} G[s, Cy(s)]\, ds. \tag{3}$$

This functional turns out to be differentiable on H, and its gradient is the Krasnosel' skii operator $\Gamma = C' f C$.

If the operator K is positive definite in H, then $\widetilde{K} = K$ and $\widetilde{K}^{1/2} = K^{1/2}$. In this case the problem of bifurcation points of equation (1) reduces to the problem of bifurcation points of the equation $\Gamma y = \lambda y$ with the Krasnosel' skii operator Γ, acting in H. If, however, K has a finite number of negative eigenvalues, then the problem reduces to the problem of bifurcation points of the equation $J\Gamma y = \lambda y$. This is also true for asymptotic bifurcation points.

2. Suppose that

$$f(s, u) = a_0(s)u + h(s, u)u, \tag{4}$$

where $a_0(s) \in E'/E$, and $h(s, u)$ ($h(s, 0) = 0$) is a function satisfying the Carathéodory conditions and defining the superposition operator $hx = h[s, x(s)]$, acting from E into E'/E. By K_0 denote the linear integral operator with kernel $k(t,s)a_0(s)$.

Assume that one of the following conditions is fulfilled:

a) the operator K is completely continuous;

b) the operator f is w-bounded (or the operator G is w-bounded, and the operator f is uniformly continuous), $a_0(s) \in (E'/E)^0$, and the operator h is continuous at the zero point.

c) $E = M_{u_0}$.

If any of these conditions is fulfilled, the Krasnosel' skii operator $\Gamma = C'fC$ has as its potential the weakly continuous and smooth (uniformly differentiable on each pair) Golomb functional Φ. Moreover, the operator Γ has at the zero point the completely continuous Fréchet derivative $B_0 = C'A_0C$, where A_0 is the operator of multiplication by the function $a_0(s)$. Therefore, using the theorem of M. A. Krasnosel' skii (^{2}) and the theorem from (^{3}), we obtain the following statements.

Theorem 1. Let the operator K be positive definite and $a_0(s) \neq 0$. Then every eigenvalue of the operator K_0 is a bifurcation point for equation (1).

Theorem 2. Let the operator K have a finite number of negative eigenvalues and $a_0(s) = c \neq 0$. Then for $c < 0$ every positive, and for $c > 0$ every negative, eigenvalue of the operator K_0 is a bifurcation point for equation (1).

3. Suppose now that

$$f(s,u) = a_\infty(s)u + \omega(s,u), \tag{5}$$

where $a_\infty(s) \in E'/E$, and $\omega(s,u)$ is a function satisfying the Carathéodory conditions, with

$$\lim_{u\to\infty} \text{as} \frac{\omega(s,u)}{u} = 0, \tag{6}$$

$$|\omega(s,u)| \leq a(s) + b(s)|u|, \tag{7}$$

where $a(s) \in E'$, $b(s) \in E'/E$ (under these assumptions, obviously, the superposition operator $\omega x = \omega[s, x(s)]$ acts from E into E'). Denote by K_∞ the linear integral operator with kernel $k(t,s)a_\infty(s)$.

Assume that one of the following conditions is satisfied:

a) the operator K is completely continuous;

b) the operator f is w-bounded (or the operator G is w-bounded, and the operator f is uniformly continuous); $a_\infty(s) \in (E'/E)^0$; $b(s) \in (E'/E)^0$;

c) $E = M_{u_0}$.

Each of these conditions guarantees that the Krasnosel' skii operator Γ has a weakly continuous and smooth potential—the Golomb functional Φ. Moreover, under any of these conditions the operator Γ has the completely continuous asymptotic derivative $B_\infty = C' A_\infty C$, where A_∞ is the operator of multiplication by the function $a_\infty(s)$. Therefore, using somewhat modified results from (3), we obtain the following assertions.

Theorem 3. *Let the operator K be positive definite and $a_\infty(s) \neq 0$. Then every eigenvalue of the operator K_∞ is a bifurcation point for equation (1).*

Theorem 4. *Let the operator K have a finite number of negative eigenvalues and let $a_\infty(s) = c \neq 0$. Then for $c < 0$ every positive eigenvalue, and for $c > 0$ every negative eigenvalue of the operator K_∞, is a bifurcation point for equation (1).*

4. Let the superposition operator $fx = f[s, x(s)]$ act from one ideal space E_1 into another E_2. In the case of a regular (see (5)) space E_2, the requirement that it be w-bounded is equivalent to the relation

$$\lim_{\operatorname{mes} D \to 0} \sup_{\|x\| \leq r} \|P_{Dfx}\| = 0 \tag{8}$$

(P_D is the operator of multiplication by the characteristic function of the set $D \subset \Omega$), and the requirement of uniform continuity is equivalent to the relation

$$\lim_{\operatorname{mes} D, \rho \to 0} \sup_{\|x\|, \|y\| \leq r;\ \|x-y\| \leq \rho} \|P_{Dfx} - P_{Dfy}\| = 0. \tag{9}$$

Therefore, in the case under consideration, uniform continuity is a weaker restriction. If the space E_2 does not possess the regularity property, the requirements of w-boundedness and uniform continuity are independent.

Consider, in particular, the case when E_1 is the Orlicz space $\mathcal{L}_M$, generated by the N-function $m(s, u)$, and E_2 is the Orlicz space $\mathcal{L}_N$, generated by the N-function $n(s, u)$.

Lemma 1. *The superposition operator $fx = f[s, x(s)]$ acting from $\mathcal{L}_M$ into $\mathcal{L}_N$ is w-bounded if there exists a function $\Psi(u)$ satisfying the condition*

$$\lim_{u \to \infty} \frac{\Psi(u)}{u} = \infty, \tag{10}$$

such that for every $r > 0$ one can specify $a_r \in \mathcal{L}_1$, $b_r, \lambda_r > 0$, such that

$$\Psi\{n[s, \lambda_r f(s, u)]\} \leq a_r(s) + b_r m(s, u/r). \tag{11}$$

Lemma 2. The superposition operator acting from $\mathcal{L}_m$ to $\mathcal{L}_N$, $fx = f[s, x(s)]$, is uniformly continuous if and only if, for some $c > 0$ and arbitrary $\varepsilon, r > 0$, one can specify $a_{\varepsilon,r} \in \mathcal{L}_1$, $b_{\varepsilon,r} > 0$ such that

$$n\left[s, \frac{f(s,u) - f(s,v)}{\varepsilon}\right] \leq a_{\varepsilon,r}(s) + m\left[s, b_{\varepsilon,r}(u-v)\right] + c\left[m\left(s, \frac{u}{r}\right) + m\left(s, \frac{v}{r}\right)\right]. \tag{12}$$

Lemmas 1 and 2 also cover the cases when the functions $n(s,u)$ do not satisfy the Δ_2-condition.

5. In conclusion, we note that the results of the paper carry over to equations with a Lebesgue integral with respect to an arbitrary measure, in particular to infinite systems; in this case, instead of Banach spaces, one may consider locally convex spaces.

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Received
28 IV 1969

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Note: Figure translations are in progress. See original paper for figures.

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