APPLICATION OF THE METHOD OF SUCCESSIVE APPROXIMATIONS TO THE INTEGRATION OF BOUNDARY-LAYER EQUATIONS

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Abstract

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É. A. Kovach, G. A. Tirskii

APPLICATION OF THE METHOD OF SUCCESSIVE APPROXIMATIONS TO THE INTEGRATION OF BOUNDARY-LAYER EQUATIONS

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The method of successive approximations, in one form or another, has repeatedly been used to prove existence theorems and to compute solutions of the system of Prandtl boundary-layer equations ([1–3]). In the present paper a new variant of the method of successive approximations is proposed for the numerical integration of two-dimensional equations of an asymptotically thin boundary layer with an arbitrary pressure gradient, in the case where a solution of the problem exists. The successive approximations can be obtained both analytically and numerically. Implementation of the algorithm in the first case gives approximate solutions in explicit form. Subsequent, more distant approximations can practically be obtained numerically and provide refinement. In this sense, provided there is convergence—which was verified by direct computation of a large number of iterations—the method may be regarded as exact.

Some ideas are contained in the approximate methods of S. M. Targ ([4]) and E. I. Shvets ([5]), which replace the exact formulation of the problem by an approximate one, introducing the concept of a layer of finite thickness and thereby requiring the introduction of additional boundary conditions not following from the formulation of the problem. In these works, practically only the first approximation is computed (the second according to Shvets). In this connection, the theoretical and practical possibility of computing subsequent approximations remains unclear. L. G. Loitsyanskiy showed ([6]) that, in this form, this method coincides in its results with the approximate linearized one-parameter method.

The fundamental difference of the present approach consists in constructing the iterations in such a way that the $(n+1)$-st approximation for the exact formulation of the asymptotic boundary-layer problem can be written in recurrent form through the n-th one, and thereby a new numerical scheme is obtained, according to which the solution can be computed with any desired degree of accuracy. This is achieved by first integrating the momentum equation exactly twice with respect to the normal coordinate, after which a function $\delta(s)$, controlled by the external boundary condition, is introduced, and then the general iterative process is constructed in explicit form at each step.

1. The dynamic equations of a two-dimensional asymptotically thin boundary layer for an incompressible fluid can be reduced to a single equation ([6])

$$\frac{\partial^3\varphi}{\partial\eta^3} + (\varphi + 2\xi\varphi_\xi)\frac{\partial^2\varphi}{\partial\eta^2} = \beta(\xi)\left[\left(\frac{\partial\varphi}{\partial\eta}\right)^2 - 1\right] + 2\xi\frac{\partial\varphi}{\partial\eta}\frac{\partial^2\varphi}{\partial\eta\partial\xi} \tag{1}$$

with boundary conditions

$$\varphi(\xi,0) + 2\xi\varphi_\xi(\xi,0) = f_w(\xi), \qquad \partial\varphi(\xi,0)/\partial\eta = 0, \qquad \partial\varphi(\xi,\infty)/\partial\eta = 1. \tag{2}$$

The initial data are obtained from the solution of equation (1) at $\xi = 0$ with the same boundary conditions (2). Let us integrate equation (1) with respect to η from η to ∞.

$$\partial^2\varphi/\partial\eta^2 = (1-\tilde{u})\varphi + \tilde{\Delta}^{**} + \beta(\xi)(\tilde{\Delta}^* + \tilde{\Delta}^{**}) + 2\xi\left[(1-\tilde{u})\partial\varphi/\partial\xi + \partial\tilde{\Delta}^{**}/\partial\xi\right],$$

$$\tilde{u} = \frac{\partial\varphi}{\partial\eta}, \qquad \tilde{\Delta}^* = \int_\eta^\infty (1-\tilde{u})\,d\eta', \qquad \tilde{\Delta}^{**} = \int_\eta^\infty (1-\tilde{u})\tilde{u}\,d\eta'. \tag{3}$$

Introduce into equation (3) new independent variables

$$s = \xi, \qquad z = \eta\delta^{-1/2}(s), \tag{4}$$

where $\delta(s)$ is as yet an undetermined function* of s. Then for $u(s,z) = \tilde{u}(\xi,\eta)$ we obtain the equation

$$\partial u/\partial z = F[u(s,z);\delta(s)], \tag{5}$$

where

$$F[u(s,z);\delta(s)] = [\delta(s)+s\delta'(s)][(1-u)f+\Delta^{**}]+\beta(s)\delta(s)(\Delta^*+\Delta^{**})+2s\delta(s)\left[(1-u)\partial f/\partial s + \partial\Delta^{**}/\partial s\right]+\sqrt{\delta(s)}(1-$$

$$f(s,z) = \int_0^z u(s,z')\,dz', \qquad \Delta^*(s,z) = \int_z^\infty (1-u)\,dz', \qquad \Delta^{**}(s,z) = \int_z^\infty u(1-u)\,dz'.$$

The right-hand side of equation (5) depends on the as yet undetermined function $\delta(s)$. For a given $u(s,z)$, define δ as the solution of the first-order ordinary differential equation

$$\int_0^\infty F[u(s,t);\delta(s)]\,dt = 1, \tag{6}$$

which follows after integrating (5) and satisfying the boundary condition $u(s,\infty)=1$. The initial condition for this equation is found from the algebraic relation obtained from equation (6) itself, if in it one sets $s=0$. We note that this initial condition determines the unique bounded solution of equation (6) for sufficiently small s. To solve the partial differential equation (5) and the ordinary equation (6) for $\delta(s)$ coupled with it, an iterative process is proposed. Equation (5) is equivalent to the integral equation

$$u(s,z) = \int_0^z F\,dz' = (\delta + s\delta')A(s,z) + [\beta(s)B(s,z) + 2sC(s,z)]\delta +$$
$$+\sqrt{\delta}\,f_w(s)\int_0^z (1-u)\,dz' \tag{7}$$

and relation (6), which for a given function $u(s,z)$ is an ordinary differential equation with respect to $\delta(s)$,

$$(\delta+s\delta')A(s,\infty)+[\beta(s)B(s,\infty)+2sC(s,\infty)]\delta+\sqrt{\delta}\,f_w(s)\int_0^\infty (1-u)\,dz' = 1, \tag{8}$$

where

$$A(s,z) = \int_0^z [(1-u)f + \Delta^{**}]\,dz', \qquad B(s,z) = \int_0^z (\Delta^* + \Delta^{**})\,dz',$$
$$C(s,z) = \int_0^z \left[(1-u)\frac{\partial f}{\partial s} + \frac{\partial \Delta^{**}}{\partial s}\right] dz'.$$

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* This function is not the boundary-layer thickness, since below the boundary conditions are satisfied as $\eta \to \infty$, and not at a finite thickness associated with $\delta(s)$.

From practical information about the qualitative character of the velocity profile in the boundary layer, the zeroth approximation is chosen as $u(s,z) = u_0(s,z)$.

Then equation (8) is solved. Next, from (7), $u_1(s,z)$ is found, and the process is then repeated. From the results of the last iteration we compute the dimensionless wall friction

$$\tau(s) = f_\eta(\xi,0) = \left(\frac{\partial u(s,0)}{\partial z}\right)\delta^{-1/2}$$

from the right-hand side of equation (5) at $z=0$. We give the final form, more convenient for numerical iterations, of the algorithm indicated above:

$$s\delta_n' + \left[1 + \beta(s)\frac{B_n(s,\infty)}{A_n(s,\infty)} + 2s\frac{C_n(s,\infty)}{A_n(s,\infty)}\right]\delta_n + \frac{f_w(s)\delta_n^*(s)}{A_n(s,\infty)}\sqrt{\delta_n} = \frac{1}{A_n(s,\infty)},$$

$$\begin{aligned} u_{n+1}(s,z) = {} & \frac{A_n(s,z)}{A_n(s,\infty)} + \left\{\beta\left[B_n(s,z) - \frac{A_n(s,z)}{A_n(s,\infty)}B_n(s,\infty)\right]\right. \\ & \left. +2s\left[C_n(s,z) - \frac{A_n(s,z)}{A_n(s,\infty)}C_n(s,\infty)\right]\right\}\delta_n \\ & + \left\{\left[1 - \frac{A_n(s,z)}{A_n(s,\infty)}\right]\delta_n^* - \Delta_n^*\right\} f_w(s)\sqrt{\delta_n}, \end{aligned} \tag{9}$$

$$\begin{aligned} \tau_n(s) = {} & \frac{\delta_n^{**}}{\sqrt{\delta_n}\,A_n(s,\infty)} + \beta(s)\left[\delta_n^* + \delta_n^{**} - \delta_n^{**}\frac{B_n(s,\infty)}{A_n(s,\infty)}\right]\sqrt{\delta_n} \\ & +2s\left[\frac{d\delta_n^{**}}{ds} - \delta_n^{**}\frac{C_n(s,\infty)}{A_n(s,\infty)}\right]\sqrt{\delta_n} + \left[1 - \frac{\delta_n^*\delta_n^{**}}{A_n(s,\infty)}\right] f_w(s), \end{aligned}$$

$$\delta^*(s) = \Delta^*(s,0), \qquad \delta^{**}(s) = \Delta^{**}(s,0),$$

where the index n means that the corresponding quantity must be computed from its own formula, into which $u_n(s,z)$ is to be substituted in place of $u(s,z)$.

2. An important advantage of the iterative process described is the possibility of obtaining the first approximations in explicit form, giving a solution with high accuracy when the initial function $u_0(s,z)$ is chosen successfully. A successful choice of the initial function is practically always possible, since the qualitative character of the flow in the boundary layer is often known. On the other hand, the simplicity of implementing this algorithm on a computer makes it possible to compute any number of approximations if the corresponding choice of $u_0(s,z)$ is difficult. The extension of the method to more complicated problems of boundary-layer theory is obvious.

3. **Plate:** $\beta = 0,\ f_w = 0$

$$\delta_n = A_n^{-1}(\infty), \qquad u_{n+1}(z) = \delta_n\int_0^z \left[(1-u_n)f_n + \Delta_n^{**}\right]dz', \qquad \tau_{n+1} = \delta_n^{**}\sqrt{\delta_n}.$$

Choosing the initial function in the form $u_0 = 1 - \exp(-z)$, we easily obtain in the first approximation:

$$\delta_0 = 0.8, \qquad u_1 = 1 - 0.8(1+z)\exp(-z) - 0.2\exp(-2z), \qquad \tau_1 = 0.4472.$$

Figure 1 and Figure 2

Figure 1: Figure 1 and Figure 2

Higher-order iterations refine these results:

$$\tau_2 = 0.4581, \qquad \tau_3 = 0.4634, \qquad \tau_4 = 0.4705.$$

The exact value is 0.4696. The third approximation gave $u_3(\eta\delta_3^{-1/2})$, coinciding with the Blasius curve to graphical accuracy. If the probability integral $u_0 = \Phi(z)$ is taken as the initial function, then we obtain $\tau_1 = 0.4673$.

Cylinder: $\beta(s) = (1-2s)(1-s)^{-1}$, $f_w = 0$. Two initial functions were considered: $u_0 = 1 - \exp(-z)$ and $u_0 = \Phi(z)$. The prescribed relative accuracy $\varepsilon = 0.001$ was achieved with the first function only in the fourth approximation, and with the second already in the second approximation. The second approximation in both cases is written out in explicit form. For lack of space we write out only the first approximation:

$$\delta_0(s) = 4s^{-(2+2/\pi)}(1-s)^{-(1+2/\pi)} \int_0^s t^{1+2/\pi}(1-t)^{1+2/\pi}\,dt, \qquad \delta_0(0) = \frac{2\pi}{\pi+1},$$

$$u_1(s,z) = a_1(z) + \beta(s)\delta_0(s)b_1(z),$$

$$\tau_1(s) = 4\,\frac{\sqrt{2}-1}{\sqrt{\pi\delta_0(s)}} + \frac{\beta(s)\sqrt{\delta_0(s)}}{\sqrt{\pi}}\left[1 - \frac{2}{\pi}\left(\sqrt{2}-1\right)\right],$$

where the functions $a_1(z)$ and $b_1(z)$ are expressed in a known way in terms of elementary functions. Next, 10 approximations were computed: $\tau_1 = 1.2509$; 1.2414; 1.2308; 1.2322; 1.2311, 1.2304, 1.2316, 1.2308, 1.2313, 1.2309. The position of the separation point: $s_1^* = 0.6349$, 0.6134, 0.6130, 0.6126,

Fig. 1. 1—$\tau_1(s)$; 2—$\tau_2(s)$; 3—$\tau_3(s)$

Fig. 2. 1—u_0; 2—u_1; 3—u_2; 4—u_3,
where $z_n = \eta\delta_n^{-1/2}(s)$

0.6120, 0.6125, 0.6123, 0.6126, 0.6124, 0.6125. The character of the approximations to the separation point is one-sided. In Fig. 1, $\tau(s)$ is shown for 3 approximations, and in Fig. 2—the velocity profiles in various sections for 3 approximations.

Moscow Institute of Physics and Technology

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Note: Figure translations are in progress. See original paper for figures.