



Soviet-era science, translated into English

Reports of the Academy of Sciences of the USSR

A. Ya. ISMAILOV, M. A. TAGIEVA

1970

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-197001.13166>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1970. Volume 195, No. 5

MATHEMATICS

A. Ya. ISMAILOV, M. A. TAGIEVA

ON THE REPRESENTATION OF GENERALIZED ANALYTIC FUNCTIONS BY SERIES OF PSEUDOPOLYNOMIALS

(Presented by Academician I. N. Vekua on 1 VI 1970)

Generalized analytic functions in the sense of I. N. Vekua are solutions of an equation of the form

$$\partial_{\bar{z}}W + AW + B\bar{W} = F, \quad (1)$$

where $A, B, F \in L_p$, $p > 2$.

Solutions of these equations preserve a number of basic topological properties of analytic functions of one complex variable (the uniqueness theorem, the argument principle, etc.). Such analytic facts as Taylor and Laurent expansions, the Cauchy integral formula, etc., extend to these solutions. (For a detailed exposition see ⁽¹⁾.)

In this article we consider questions of interpolation of generalized analytic functions by generalized polynomials. In his monograph I. N. Vekua first drew attention to the possibility of interpolating generalized analytic functions and indicated a method for constructing solutions of equation (1) that assume prescribed values at fixed points.

As is known, in the study of the convergence of interpolation processes the integral form of the remainder term is of great help.

In this article we give analogues of the Lagrange-Hermite and Abel-Goncharov formulas for a particular kind of generalized analytic functions, i.e., for functions that are solutions of equations of the form

$$\partial_{\bar{z}}W + AW + B\bar{W} = 0, \quad (2)$$

where $A, B \in L_{p,2}$, $p > 2$ ⁽¹⁾, and with their aid we prove several convergence theorems.

The theory of solutions of equation (2), which is a special case of equation (1) and is known in the literature under the name of the theory of pseudoanalytic functions, was the subject of investigations by L. Bers ⁽²⁾, who proceeded from other ideas.

In establishing the Lagrange-Hermite and Abel-Goncharov formulas we use the integration process for pseudoanalytic functions introduced by L. Bers and the Cauchy formula ⁽²⁾.

We introduce the following notation ⁽³⁾: a set B of type \mathfrak{M} is a closed set containing at least two points, whose complement in the extended plane is a simply connected domain containing the infinitely distant point; $\overline{CB} = B'$; L is the boundary of B ; $V = \varphi(z)$ is a function conformally mapping the domain B' onto $|V| > 1$, $\varphi(\infty) = \infty$, $\varphi'(\infty) > 0$, $[\varphi'(\infty)]^{-1} = c$; $z = \psi(V)$ is the inverse function to $\varphi(z)$; L_ρ , $\rho > 1$, are level lines of B' ; B_ρ is the domain bounded by L_ρ ; $\{z_k^{(n)}\}_{k=0}^\infty$, $z_k^{(n)} \in \overline{B}$, are interpolation nodes;

$$\omega_n(z) = \prod_{k=1}^{n+1} (z - z_k^{(n)}); \quad M_n(\rho) = \sup_{z \in L_\rho} |\omega_n(z)|, \quad \rho > 1,$$

$$M_n = \sup_{z \in L} |\omega_n(z)|.$$

Consider the interpolation problem: it is required to construct a pseudopolynomial $P_n(z)$ of degree n , which at the prescribed $n + 1$ points z_1, z_2, \dots, z_{n+1} would take prescribed values, i.e.,

$$P_n(z_k) = W(z_k).$$

In papers ^(1, 2) it is proved that such a polynomial exists and is unique. For the interpolation polynomial $P_n(z)$ the following holds.

Theorem 1. Let B be a domain bounded by a closed piecewise-smooth Jordan curve L . Let the function $W(z)$ be pseudoanalytic in B and continuous in $B + L$, and let the interpolation nodes z_k ($k = 1, \dots, n + 1$) (not necessarily distinct) lie in B .

Then the interpolation polynomial $P_n(z)$, satisfying the conditions $P_n(z_k) = W(z_k)$ ($k = 1, \dots, n + 1$), can be represented by the formula

$$P_n(z) = \int_L Z^{(-1)} \left[\frac{\omega(\zeta) - \omega(z)}{\omega(\zeta)} \frac{iW(\zeta)}{2\pi} d\zeta, \zeta; z \right], \quad \omega(z) = \prod_{k=1}^{n+1} (z - z_k). \quad (3)$$

Moreover, for the remainder term of the interpolation the relation

$$R_n(z) = \int_L Z^{(-1)} \left[\frac{\omega(z)}{\omega(\zeta)} \frac{iW(\zeta)}{2\pi} d\zeta, \zeta; z \right]. \quad (4)$$

Formula (2) makes it possible to obtain the following estimate of the remainder term.

Theorem 2. If $B + L$ is a set of type \mathfrak{M} and $W(z)$ is a function regular pseudoanalytic in \overline{B}_{ρ_0} , $\rho_0 > 1$, then for the remainder term of interpolation the estimate

$$|W(z) - P_n(z)| \leq KM(\rho_0) \frac{\rho_0 + r}{\rho_0 - r} \frac{\sup_{z \in L_r} |\omega_n(z)|}{\inf_{z \in L_{\rho_0}} |\omega_n(z)|}, \quad |z| \in \overline{B}_r, \quad 1 \leq r < \rho_0,$$

holds, where K is a constant depending on the generating pair (F, G) , and

$$M(\rho_0) = \frac{1}{2\pi} \int_0^{2\pi} |W[\psi(\rho e^{i\theta})]| d\theta.$$

With the aid of this estimate one proves

Theorem 3. In order that the sequence of interpolation pseudopolynomials $\{P_n(z)\}$, for any function regular pseudoanalytic on a set \overline{B} of type \mathfrak{M} , converge uniformly to $W(z)$, it is necessary and sufficient that, as $n \rightarrow \infty$,

$$\left| \sqrt[n+1]{\omega_n(z)/c\varphi(z)} \right| \equiv |\theta_n(z)| \rightarrow 1 \quad (5)$$

in B' , or else

$$\lim_{n \rightarrow \infty} \frac{\sqrt[n+1]{M_n}}{c} = 1. \quad (6)$$

If the function $W(z)$ is regular pseudoanalytic in B_{ρ_0} , $\rho_0 > 1$, and one of the conditions (3) or (4) is fulfilled, then for any ρ_1 , $1 < \rho_1 < \rho_0$,

$$|W(z) - P_n(z)| \leq M(\rho, k)/\rho_1^{n+1}, \quad z \in \overline{B},$$

where $M(\rho_1, K)$ is a constant independent of n , but dependent on the generating pair (F, G) and ρ_1 .

Now consider the Abel-Goncharov interpolation problem ⁽⁴⁾: suppose we are given a pseudoanalytic function $W(z)$, defined in some domain D , and $\{\lambda_n\}$

is an arbitrary sequence of complex numbers lying in this domain. Find a pseudopolynomial $Q_n(z)$ of degree n having the property that

$$Q^{(k)}(\lambda_k) = W^{(k)}(\lambda_k). \quad (7)$$

Here the derivatives are understood in the sense of L. Bers. By Bers' theorem ⁽²⁾ such a polynomial exists and is unique. Let us clarify the circumstances under which $Q_n(z)$ converges to $W(z)$. We first state the following theorem, which gives an analogue of the Abel-Goncharov interpolation formula for pseudoanalytic functions.

Theorem 4. Let $W(z)$ be a pseudoanalytic function defined in a domain D , let $\{\lambda_n\}$ be an arbitrary sequence of complex numbers lying in this domain, and let $\{(F_k, G_k)\}_0^\infty$, $(F_0, G_0) = (F, G)$, be a sequence of generating pairs.

Then the interpolation pseudopolynomial satisfying condition (5) is determined as follows:

$$Q_n(z) = \sum_{k=0}^n [\Phi(\lambda_k)P_k^*(z) + \Psi(\lambda_k)P_k^{**}(z)],$$

where

$$P_k^*(z) = \int_{\lambda_0}^z \cdots \int_{\lambda_{k-1}}^{z_k} F_k(z_k) d_{(F_1, G_1)} z_1 \cdots d_{(F_k, G_k)} z_k,$$

$$P_k^{**}(z) = \int_{\lambda_0}^z \cdots \int_{\lambda_{k-1}}^{z_k} G_k(z_k) d_{(F_1, G_1)} z_1 \cdots d_{(F_k, G_k)} z_k,$$

and the remainder term $R_n(z) = W(z) - P_n(z)$ is determined by the formula

$$R_n(z) = \int_{\lambda_0}^z \cdots \int_{\lambda_n}^{z_n} W^{(n+1)}(z_{n+1}) d_{(F_1, G_1)} z_1 \cdots d_{(F_{n+1}, G_{n+1})} z_{n+1}.$$

Theorem 5. Let $\lim_{n \rightarrow \infty} \lambda_n = \lambda$, and let $W(z)$ be a pseudoanalytic function in some disk $|z| < R$ with respect to a sequence of generating pairs $\{(F_n, G_n)\}_0^\infty$, $(F_0, G_0) = (F, G)$.

Then, if

$$1) \sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty; \quad 2) \lim_{n \rightarrow \infty} \sqrt[n]{\Pi(2M_k/m_k)^2} \leq M; \quad 3) \lim_{n \rightarrow \infty} \sqrt[n]{c_n} \leq c,$$

then the series

$$\sum_{n=0}^{\infty} \Phi(\lambda_n) P_n^*(z) + \Psi(\lambda_n) P_n^{**}(z)$$

converges uniformly to the pseudoanalytic function $W(z)$ inside the disk $|z| < R/CM$, where

$$M_k = \max_{z \in D} \{|F_k|, |G_k|\}, \quad m_k = \min_{z \in D} |F_k \overline{G_k} - \overline{F_k} G_k|,$$

and c_n is a positive constant depending on the generating pair (F, G) .

Azerbaijan State University
named after S. M. Kirov
Baku

Received
27 IV 1970

CITED LITERATURE

- ¹ I. N. Vekua, *Generalized Analytic Functions*, Moscow, 1959.
- ² L. Bers, *Theory of pseudo-analytic functions*, Lecture notes, N.-Y., 1953.
- ³ V. I. Smirnov, N. A. Lebedev, *Constructive Theory of Functions*, Moscow, 1964.
- ⁴ M. A. Evgrafov, *The Abel-Goncharov Interpolation Problem*, Moscow, 1954.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.