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Abstract

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MATHEMATICS

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ON THE UNIQUENESS OF THE SOLUTION OF THE TRICOMI PROBLEM FOR ONE CLASS OF NONLINEAR EQUATIONS

(Presented by Academician M. A. Lavrent' ev on 2 VI 1969)

Let D be a simply connected bounded domain in the plane of the variables x, y with boundary $\Gamma = \sigma + AC + CB$, where σ is an open Jordan curve in the half-plane $y > 0$ with endpoints at $A(0,0)$, $B(1,0)$. The curves AC and CB lie in the lower half-plane and are given by the equations

$$x - \frac{2}{m+2}(-y)^{(m+2)/2} = 0, \qquad x + \frac{2}{m+2}(-y)^{(m+2)/2} = 1$$

respectively. The number m is odd and positive.

In this domain we consider the equation

$$T[u] \equiv y^m u_{xx} + u_{yy} = f(x, y, u), \tag{1}$$

where $f(x, y, u)$ is a function which, for all $(x, y) \in \overline{D}$ and for all $|u| < \infty$, satisfies the following conditions: $f(x, y, u)$ and $f_u(x, y, u)$ are continuous,

$$f_u(x, y, u) > 0, \tag{2}$$

$$f_u(x, y, u) \le m(m+4)4^{-(m+4)/(m+2)}(m+2)^{-2m/(m+2)}y^{-2} \quad \text{for } y < 0. \tag{3}$$

Equation (1) is elliptic for $y > 0$, degenerates parabolically for $y = 0$, is hyperbolic for $y < 0$, and the curves AC and CB are its characteristics.

By $\varphi(s)$ we denote a continuous function prescribed on σ of the arc length s, measured from the point $B(1,0)$ in the positive direction. The function $\psi(x, y)$

is defined for $(x, y) \in \overline{D},\ y \leq 0$ and is continuously differentiable once. In addition, we introduce the notation $D_1 = D \cap \{y > 0\}$; $D_2 = D \cap \{y < 0\}$. By D^ε we denote the subdomain of D obtained by cutting off from D a piece by the curves σ_ε and $B_\varepsilon C_\varepsilon$, where

$$\sigma_\varepsilon:\ (x-1)^2 + \frac{4}{(m+2)^2} y^{m+2} = \varepsilon^2, \qquad B_\varepsilon C_\varepsilon:\ x + \frac{2}{m+2}(-y)^{(m+2)/2} = 1 - \varepsilon.$$

Accordingly, we shall denote

$$D_1^\varepsilon = D^\varepsilon \cap \{y > 0\}, \qquad D_2^\varepsilon = D^\varepsilon \cap \{y < 0\}.$$

Tricomi problem. It is required to find a solution $u(x, y)$ of equation (1), regular in D and continuous in $\overline{D}$, satisfying the conditions

$$u|_\sigma = \varphi, \tag{4}$$

$$u|_{AC} = \psi|_{AC}, \tag{5}$$

and moreover its first derivatives may have integrable singularities at the point A and on the closed arc BC.

Theorem 1. *If conditions (2), (3) are satisfied, then problem (1), (4), (5) can have only one solution.*

The validity of this assertion follows directly from the following theorem.

Theorem 2. *If conditions (2), (3) are satisfied, then any solution of problem (1), (4), (5), if it exists, satisfies the estimate*

$$|u| \leq kK, \tag{6}$$

where $k > 0$ is a constant depending on the diameter of the domain D, and the number K is determined

is determined as follows:

$$K = \max_{\overline{D}} |f(x, y, 0)| + \max_{\overline{\sigma}} |\varphi(s)| + \max_{\overline{AC}} |\psi(x, y)| + \max_{\overline{AC}} |x^{m/(m+2)} \psi_\eta|, \tag{7}$$

where

$$\psi_\eta = -\frac{1}{2}\psi_x + \frac{1}{2}(-y)^{-m/2}\psi_y.$$

First consider the equation

$$T[u] = -c(x,y)u + F(x,y) \tag{8}$$

and prove the validity of estimate (6) for it.

Theorem 3. *If the function $z(x,y)$ satisfies the conditions*

$$z(x,y) \in C(\overline{D}) \cap C^1(D \cup AC) \cap C^2(D),$$

$$T[z] + C(x,y)z = E(x,y) \geq 0, \tag{9}$$

$$z_\eta(x,y)\big|_{AC} \geq 0, \tag{10}$$

then the greatest positive value of the function $z(x,y)$, if it exists, is attained on $\overline{\sigma}$.

To prove this assertion we shall use the following known theorems.

Theorem (see (1, 2)). If conditions (2), (3), (9), (10) are satisfied, then the function $z(x,y)$ can attain its greatest positive value in the closed domain $\overline{D}^\varepsilon$ only on the segment $y = 0,\ 0 \leq x \leq 1-\varepsilon$.

Theorem (see (2)). Under the conditions of the preceding theorem, if $z(x,y)$ attains its greatest positive value in $\overline{D}_2$ at the point $(x_0, 0)$, $0 < x_0 < 1$, then

$$z_y(x_0, 0) > 0. \tag{11}$$

Proof of Theorem 3. Fix an arbitrary number $\varepsilon > 0$. According to the condition of the theorem, $z(x,y) \in C(\overline{D})$, and therefore it is continuous in the closed subdomain $\overline{D}^\varepsilon$. Denote its greatest positive value in $\overline{D}^\varepsilon$ by M_ε. According to the first theorem, we have

$$0 \leq \max_{\overline{D}^\varepsilon 2} z(x,y) = \max_{\overline{AB}_\varepsilon} z(x,y) = z(x_0, 0), \qquad 0 \leq x_0 \leq 1-\varepsilon,$$

where, naturally, $z(x_0, 0) \leq M_\varepsilon$. By the known maximum principle for elliptic equations (see (3)) we have

$$\max_{\overline{D}^\varepsilon 1} z(x,y) = \max_{\overline{\sigma}_0 \cup \overline{AB}_\varepsilon \cup \overline{\sigma}_\varepsilon} z(x,y) \leq M_\varepsilon,$$

where

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$$\sigma_0 = \sigma \setminus \left\{ (x-1)^2 + \frac{4}{(m+2)^2} y^{m+2} \leq \varepsilon^2 \right\}.$$

Thus, we obtain

$$\sup \left\{ \max_{\overline{D}^{\varepsilon} 2} z(x,y),\ \max_{\overline{D}^{\varepsilon} 1} z(x,y) \right\} = \max_{\overline{D}^{\varepsilon}} z(x,y) = \max_{\overline{\sigma}_0 \cup \overline{AB}_{\varepsilon} \cup \overline{\sigma}_{\varepsilon}} z(x,y). \tag{12}$$

On the other hand, $z(x,y)$ is continuous in the closed domain $\overline{D}_1$, twice continuously differentiable in D_1, and satisfies condition (9). According to the maximum principle,

$$\max_{\overline{D}_1} z(x,y) = \max_{\overline{\sigma} \cup \overline{AB}} z(x,y),$$

and, consequently,

$$\max_{\overline{\sigma}_{\varepsilon}} z(x,y) \leq \max_{\overline{\sigma} \cup \overline{AB}} z(x,y). \tag{13}$$

From (12) and (13) we obtain

$$\max_{\overline{D}^{\varepsilon}} z(x,y) \leq \max_{\sigma \cup AB} z(x,y) = \max_{\overline{D} 1} z(x,y).$$

Suppose that $\max_{\overline{D}1} z(x,y)$ is attained on AB at some point $(x_0, 0)$, $0 < x_0 < 1$. Then, according to Zaremba' s lemma $({}^{2,4})$, at this point we shall have $z_y(x_0, 0) < 0$, which contradicts inequality (11). Consequently, our supposition is false. Thus, finally we obtain

$$M_{\varepsilon} = \max_{\sigma} z(x,y)$$

for arbitrary $\varepsilon > 0$, as was required to prove.

Proof of Theorem 2 for equation (8). Consider the function

$$h(y) = K(\alpha - \beta e^{2y}), \tag{14}$$

where

$$\beta > \sup \left\{ (1 + e^{2\overline{y}}),\ \left[1 + 2 \left(\frac{2}{m+2} \right)^{m/(m+2)} e^{-1} \right] \right\}, \quad \overline{y} = \max_{\overline{D}} |y|, \quad \alpha > 1 + \beta e^{2\overline{y}}.$$

By direct calculation we obtain

$$h_\eta(y) = -\beta(-y)^{-m/2} e^{2y} K,$$

$$-\left. h_\eta(y)\right|_{AC} = \beta \left(\frac{2}{m+2}\right)^{m/(m+2)} \exp\left[-2\left(\frac{m+2}{2}x\right)^{2/(m+2)}\right] x^{-m/(m+2)} K.$$

Hence, multiplying both sides by $x^{m/(m+2)}$, we have

$$-x^{m/(m+2)}[h_\eta(y)]_{AC} \geq K \geq \max_{\overline{AC}} \left| x^{m/(m+2)} \psi_\eta \right|,$$

i.e.

$$-x^{m/(m+2)}[h_\eta(y)]_{AC} \geq \left. \left| x^{m/(m+2)} \psi_\eta \right|\right._{AC},$$

whence it follows that

$$-x^{m/(m+2)}[h_\eta(y)]_{AC} \geq -x^{m/(m+2)}[\psi_\eta(x,y)]_{AC}, \tag{15}$$

$$-x^{m/(m+2)}[h_\eta(y)]_{AC} \geq x^{m/(m+2)}[\psi_\eta(x,y)]_{AC}. \tag{16}$$

Further,

$$-\{T[h] + c(x,y)h\} = \{4\beta e^{2y} - c(\alpha - \beta e^{2y})\}K \geq K \geq \max_{\overline{D}} |F(x,y)|. \tag{17}$$

In addition, the inequalities

$$\left. h\right|_\sigma \geq \max_\sigma |\varphi(s)|, \tag{18}$$

$$\left. h\right|_{AC} \geq \max_{\overline{AC}} |\psi(x,y)|. \tag{19}$$

are obvious.

Take the function $z(x,y) = u(x,y) - h(y)$, where $u(x,y)$ is a solution of problem (4), (5), (8). For it, according to (17) and (15), we have

$$T[z] + cz = F(x,y) - \{T[h] + ch\} \geq F(x,y) + \max_{\overline{D}} |F(x,y)| \geq 0, \quad \left. z_\eta(x,y)\right|_{AC} \geq 0.$$

From (18) and (19), respectively, the inequalities follow

$$z(x,y)\Big|_{\sigma}\leq 0,\qquad z(x,y)\Big|_{AC}\leq 0. \tag{20}$$

Thus, the function $z(x,y)$ satisfies all the requirements of Theorem 3. Therefore it must be that

$$0\leq\max_{\overline{D}}z(x,y)=\max_{\sigma}z(x,y),$$

whence, on the basis of inequality (20), we obtain

$$z(x,y)=u(x,y)-h(y)\leq 0\qquad\text{everywhere in }\overline{D}. \tag{21}$$

Next consider $z_1(x,y)=-u(x,y)-h(y)$, for which, analogously, we obtain $T[z_1]+cz_1\geq -F(x,y)+\max|F(x,y)|\geq 0$. On the basis of condition (16) we have $(z_1)_\eta|_{AC}\geq 0$, and from (18), (19) it follows that

$$z_1(x,y)|_{\sigma}\leq 0,\qquad z_1(x,y)|_{AC}\leq 0. \tag{22}$$

According to Theorem 3 we may write

$$z_1(x,y)=-u(x,y)-h(y)\leq 0\quad\text{everywhere in }\overline{D}. \tag{23}$$

Inequalities (22) and (23) give $u(x,y)\leq h(y)$, $-h(y)\leq u(x,y)$, whence we finally have $|u(x,y)|\leq h(y)$. Denoting $k=\max_{\overline{D}}(a-\beta e^{2y})$, we obtain estimate (6).

It is now easy to prove Theorem 2. Let there exist a solution of the Tricomi problem (1), (4), (5), namely $u(x,y)$. Substituting it into equation (1), we obtain the identity $T[u]\equiv f(x,y,u)$, which can be written in the form (5) $T[u]-f_u(x,y,\tilde{u})=f(x,y,0)$, where $\tilde{u}$ lies between zero and $u(x,y)$. In the identity thus obtained, the coefficient of $u(x,y)$ satisfies conditions (2), (3). Referring to the assertion proved above, we obtain the estimate $|u|\leq h$, which completes the proof of Theorem 2.

Suppose that the Tricomi problem (1), (4), (5) has two solutions $u_1(x,y)$ and $u_2(x,y)$. Then the difference $v(x,y)=u_1(x,y)-u_2(x,y)$ vanishes on the curves σ and AC, respectively. Consequently, $v_\eta|_{AC}=0$. Moreover, we have the identity (see (6))

$$T[v]\equiv T[u_1]-T[u_2]\equiv f(x,y,u_1)-$$
$$-f(x,y,u_2)=(u_1-u_2)\int_0^1\Phi_{u_1}(x,y,u_1,t)\,\frac{dt}{t},\quad\text{where }\Phi(x,y,u_1,t)=$$
$$=f(x,y,u_2+t(u_1-u_2)).$$

The coefficient of $v=u_1-u_2$ satisfies the required conditions.

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On the basis of Theorem 2 it follows that $v(x, y) = 0$ everywhere in $\overline{D}$, i.e. $u_1(x, y) \equiv u_2(x, y)$. The uniqueness theorem is proved.

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