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Abstract

Full Text

MATHEMATICS

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AN ITERATIVE METHOD FOR CONSTRUCTING POLYCUBIC SPLINE FUNCTIONS

1. Let there be a grid on the interval $[a, b]$

$$\Delta: \quad a = x_0 < x_1 < \dots < x_{n+1} = b, \tag{1}$$

at whose nodes the values of some function are prescribed

$$f: \quad f_0, f_1, \dots, f_{n+1}. \tag{2}$$

Suppose, in addition, that one of the conditions is satisfied

$$\text{a)} \;\; f'_x|_\Gamma = \varphi, \qquad \text{b)} \;\; f''_{xx}|_\Gamma = \psi \tag{3}$$

or c) f is periodic with period $b - a$. It is clear that one can also consider the case where condition a) is prescribed on one boundary and condition b) on the other.

We shall call a cubic spline function the solution of the boundary-value problem with boundary conditions a), b), or c) for the equation

$$a(x)\,\partial^4 u/\partial x^4 = 0; \tag{4}$$

$$a(x) \begin{cases} = 0, & x \in \Delta, \\ > 0, & x \notin \Delta, \end{cases} \tag{5}$$

and such that

$$u(x) \in C_2[a, b], \qquad u(x_i) = f(x_i), \qquad i = 0, \dots, n + 1. \tag{6}$$

For the numerical solution of the first boundary-value problem obtained, introduce on $[a, b]$ a grid δ containing the grid Δ. Denote the number of interior nodes of the grid δ by m, and the finite-dimensional vector spaces corresponding to the grids Δ, δ by V_n, V_m. Replacing (4) by a system of finite-difference equations, we have

$$\bar{a}\Lambda u = 0, \tag{7}$$

where $\bar{a}$ is the grid sampling of the function $a(x)$, and Λ is a positive difference 5-point analogue of the operator d^4/dx^4.

Taking into account the boundary conditions (2), equations (7) can be rewritten in matrix form as

$$A(Du - g) = 0, \tag{8}$$

where D is a pentadiagonal matrix in V_m, a finite-dimensional analogue of the operator d^4/dx^4, and A is a diagonal matrix in V_m, whose zero elements correspond to the nodes of the grid Δ.

To solve the system of linear algebraic equations (5), consider the iterative process

$$u^{k+1} = u^k - \tau(Lu^k - Ag), \tag{9}$$

where $L = AD$ is a nonnegative matrix in V_m, and τ is the iteration parameter.

In the known way, the iterative process (9) is reduced to the corresponding iterative process with a positive matrix in the $(m-n)$-dimensional space V_{m-n}, where V_{m-n} is the orthogonal complement of V_n in V_m. The convergence of the process is proved in (3).

Essential for the iterative process (9) is the choice of the vector of the initial approximation $u^0 = \{u_0^0, \dots, u_{m+1}^0\}$, carried out in such a way that

$$u_{i_\alpha}^0 = f(x_{i_\alpha}), \qquad \alpha = 0, \dots, n+1 \tag{10}$$

at the points $i_\alpha \in \Delta$.

2. Let, in the domain

$$G : \{a < x < b,\ c < y < d\}, \qquad \overline{G} = G + \Gamma,$$

a rectangular grid be given,

$$\Delta : \begin{array}{l} a = x_0 < x_1 < \dots < x_{n_1} = b, \\ c = y_0 < y_1 < \dots < y_{n_2} = d, \end{array}$$

on which the values $f_{ij} = f(x_i, y_j)$ are defined.

We define a two-dimensional spline function as the solution of a boundary-value problem in the domain G, under analogous assumptions of specifying on Γ the first or second normal derivatives, or periodicity of the solution with respect to one or both variables, for the equation

$$a(x,y)\,[\partial^4 u/\partial x^4 + \partial^4 u/\partial y^4] = 0, \tag{11}$$

where

$$a(x,y) \begin{cases} = 0, & (x,y) \in \Delta, \\ > 0, & (x,y) \notin \Delta, \end{cases} \tag{12}$$

under the condition

$$u(x) \in C_2(\overline{G}), \qquad u(x_i, y_i) = f_{ij}. \tag{13}$$

Introducing, analogously to item 1, a two-dimensional rectangular grid δ, the spaces $V_{n_1 n_2}, V_{m_1 m_2}$, and replacing (11) by a system of finite-difference equations, we arrive at a system of difference equations in the space $V_{m_1 m_2}$

$$\Lambda u = \bar{a}[\Lambda_1 u + \Lambda_2 u] = 0, \tag{14}$$

where $\bar{a}$ is the grid sample of the function $a(x,y)$; Λ_1, Λ_2 are positive difference approximations of $\partial^4 u/\partial x^4, \partial^4 u/\partial y^4$.

For the solution, consider an iterative process in fractional steps of splitting type

$$u^{n+1/2} = u^n - \tau \bar{a} \Lambda_1 u^{n+1/2},$$

$$u^{n+1} = u^{n+1/2} - \tau \bar{a} \Lambda_2 u^{n+1} \tag{15}$$

or else of the stabilizing operator

$$B\,\frac{u^{n+1} - u^n}{\tau} = B_1 B_2\,\frac{u^{n+1} - u^n}{\tau} = \bar{a}\Lambda u^n = \bar{a}(\Lambda_1 + \Lambda_2)u^n \tag{16}$$

(see (^{2})). The proof of convergence of the iterative process (15) is analogous to item 1. The proposed method is suitable for any p-dimensional domain

$$G : \{a_i < x_i < b_i; \quad i = 1, \dots, p\}, \qquad \overline{G} = G + \Gamma$$

with an irregular grid Δ. In this case the problem consists only in choosing the function $a(x_1, \dots, x_p)$—nonnegative, sufficiently smooth, vanishing at the nodes of the grid Δ—and the vector of the initial approximation such that $u^0 = f$ at the points $i_\alpha = \{i_{\alpha_1}, \dots, i_{\alpha_p}\} \in \Delta$.

We note that the number q of subintervals of the grid δ falling within one interval of the grid Δ does not depend on the fineness of the grid. In the case of a one-dimensional cubic spline one may set $q = 4$, and then the solution of the iterative scheme (9) is the exact solution of problem (4)–(6).

A series of numerical experiments was carried out for the iterative process (15). As the domain $\overline{G}$, the square $\{0 \leq x, y \leq 1\}$ was considered. The grid Δ was taken to be uniform. As the multiplier $a(x, y)$ in (12), functions $a(x, y) = b_1(x)b_2(y)$ were taken, where $b(s)$ are functions of varying smoothness (from C_0 to C_∞) that assume zero values at the points of the one-dimensional grid Δ.

For comparison of the results with the same data, a bicubic spline function was constructed according to the procedure described in ([3], [4]).

The computations gave good agreement, although the proposed definition of a spline function is not equivalent to the definition from ([4]) already in the two-dimensional case.

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Note: Figure translations are in progress. See original paper for figures.

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