ON A QUALITATIVE INVESTIGATION OF STABILITY IN THE LARGE AND INSTABILITY FOR ONE CLASS OF AMPLITUDE-PULSE SYSTEMS

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 517.949.2

MATHEMATICS

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ON A QUALITATIVE INVESTIGATION OF STABILITY IN THE LARGE AND INSTABILITY FOR ONE CLASS OF AMPLITUDE-PULSE SYSTEMS

(Presented by Academician V. I. Smirnov on 5 V 1969)

1°. Let us consider a system of difference equations consisting of the linear part

$$\begin{gathered} z[n+1] = Az[n] + B\xi[n] + f(n, z[n]), \\ \sigma[n] = C^* z[n] \end{gathered} \tag{S}$$

and the nonlinear part

$$\xi[n] = \varphi(n, \sigma[k] \mid k = 0, 1, \dots, n) \qquad (n = 0, 1, 2, \dots), \tag{φ}$$

which we shall denote by the symbol (S, φ, f), and in the case $f \equiv 0$ by the symbol (S, φ). Here the constant matrices A, B, C, the vector-solution $z[n]$ of the system (S, φ, f), and the vectors ξ, φ, σ have dimensions, respectively, $\nu \times \nu$, $\nu \times p$, $\nu \times m$, ν, p, p, m. The vector-function $f(n, z)$ of order ν satisfies the condition

$$\lim |f|/|z| = 0$$

as $|z| \to \infty$, uniformly in n, while the vector of nonlinearities φ may depend on n and $\sigma[k]$, $0 \leq k \leq n$. All quantities in (S, φ, f) are real. Equations (S), (φ) describe the well-known amplitude-pulse systems of automatic control with nonlinear modulators.

Let the matrix A have no eigenvalues on the unit circle $|\lambda| = 1$, where λ is a complex parameter, and let the rank of the $\nu \times p\nu$ matrix

$$\|B, AB, A^2 B, \dots, A^{\nu-1} B\|$$

be equal to ν.

Introduce into consideration a real quadratic form $F(\xi, \sigma)$ of the vectors ξ and σ. We shall say that for the system (S, φ) the **condition $\{F\}$ is satisfied** if, for every solution $z[n]$ of the system (S, φ) and the corresponding $\xi[n], \sigma[n]$, one has

$$F(\xi[n], \sigma[n]) \geq 0 \qquad (n = 0, 1, 2, \ldots).$$

In the usual way, introduce the $m \times p$ matrix of transfer functions by the equality

$$\chi(\lambda) = C^*(A - \lambda I)^{-1} B,$$

where I is the identity matrix of order $\nu \times \nu$. Regarding, in $F(\xi, \sigma)$, the arguments ξ and σ as independent, extend the form $F(\xi, \sigma)$, preserving Hermitian character, to complex values of the arguments and put

$$\widetilde{F}(\lambda, \tilde{\xi}) = F(\tilde{\xi}, \tilde{\sigma}),$$

where

$$\tilde{\sigma} = -\chi(\lambda)\tilde{\xi},$$

and $\tilde{\xi}$ is an arbitrary complex p-vector. We shall say that for the system (S, φ) the **condition** $\{\widetilde{F}\}$ **is satisfied** if the quadratic form $\widetilde{F}(\lambda, \xi)$ of the vector argument ξ is negative definite for all values of λ, $|\lambda| = 1$.

Consider the following two types of behavior of solutions of the system (S, φ):

(A) There exist numbers $c > 0$, $\varepsilon > 0$ such that, for any solution $z[n]$ of the system (S, φ) and any $n \geq n_0 \geq 0$,

$$|z[n]| \leq ce^{-\varepsilon(n-n_0)} \times |z[n_0]|, \qquad |\xi[n]| \in l_2(0, \infty),$$

and, consequently, $\xi[n] \to 0$ as $n \to \infty$.

(B) In the space $\{z\}$ there exists a cone K of the form

$$z^* H_0 z < 0,$$

where $H_0 = H_0^*$ is some matrix such that, for certain numbers $c > 0$, $\varepsilon > 0$ and for any solution $z[n]$ of the system (S, φ) satisfying the condition $z[n_0] \in K$, one has

$$|z[n]| \geq ce^{\varepsilon(n-n_0)}|z[n_0]|$$

for all $n \leq n_0 \geq 0$.

It is clear that case (A) means that the system (S, φ) is exponentially stable, while in case (B) the system (S, φ) is exponentially unstable.

2°. **Theorem 1.** (I). Suppose that for the system (S, φ) the conditions $\{F\}$ and $\{\widetilde{F}\}$ are satisfied.

Then only the two above-indicated types of behavior of solutions of the system (S, φ) are possible: either (A), or (B).

(II). If for a system (S, ψ), which differs from the system (S, φ) only in the vector of nonlinearities ψ, and for the system (S, φ) the conditions $\{F\}$ and $\{\widetilde{F}\}$ are satisfied simultaneously, then both systems have the same type of behavior of solutions: either (A), or (B). In this case the numbers $\varepsilon > 0$, $c > 0$ depend only on the coefficients of the linear part of the systems (S, φ) and (S, ψ) and on the quadratic form F.

Suppose that for the system (S, φ) the matrices A, B, C, the vector of nonlinearities φ, and, generally speaking, the coefficients of the quadratic form F depend on some real parameter a, $0 \leq a \leq 1$. Denote them by $A_a, B_a, C_a, \varphi_a, F_a$. Construct the form

$$\widetilde{F}_a(\lambda, \tilde{\xi}) = F_a(\tilde{\xi}, \tilde{\sigma}),$$

where

$$\tilde{\sigma} = -C_a^*(A_a - \lambda I)^{-1} B_a \tilde{\xi}.$$

Theorem 2. Suppose that the coefficients of the form F_a, as well as A_a, B_a, C_a, depend continuously on a, and that for $0 \leq a \leq 1$ the conditions $\{F_a\}$, $\{\widetilde{F}_a\}$ are satisfied. Then for all systems (S_a, φ_a), $0 \leq a \leq 1$, either (A) or (B) holds, and the numbers $c > 0$, $\varepsilon > 0$ do not depend on a.

Theorem 3. Suppose that the system (S, φ) satisfies the conditions $\{F\}$, $\{\widetilde{F}\}$ for some form F.

(I). Let case (A) hold for the system (S, φ). Then in the space $\{z\}$ there exists an ellipsoid

$$\Phi\{z^* H_0 z < \text{const}\}$$

with matrix $H_0 = H_0^* > 0$, such that for the solutions of the system (S, φ, f) the following holds: a) from $z[n_0] \in \Phi$ it follows that $z[n] \in \Phi$ for $n \geq n_0$; b) for any solution $z[n]$ there is an $n_0 > 0$ for which $z[n_0] \in \Phi$.

(II). Let case (B) hold for the system (S, φ). Then in the space $\{z\}$ there exists a domain

$$Q\{|z| \geq \text{const},\ \ z^* H_0 z < 0\}$$

with Hermitian matrix H_0, such that for all solutions of the system (S, φ, f), if $z[n_0] \in Q$ and $n \geq n_0$, then

$$|z[n]| \geq \text{const} \cdot e^{\varepsilon(n-n_0)}, \quad \varepsilon > 0.$$

An analogous qualitative investigation of the behavior of solutions of the system (S, φ) under more stringent assumptions was carried out in the paper (5), where instability is understood in a weaker sense. Namely, in the space $\{z\}$ of states of the system there exists a vector a such that, for $z[0] = a$,

$$|z[n]| \to \infty \quad \text{as } n \to \infty.$$

3°. Theorems 1 and 2 are proved according to the scheme of the proofs of analogous Theorems 1-3 (1) concerning the investigation of differential equations. The proof of assertion (I) of Theorem 3 is analogous to the proof of Theorem 4

(2), while assertion (II) of Theorem 3 is established according to the scheme of the proof of case (B) of Theorem 1. The following lemmas are used.

Lemma 1. Let $V(z)$ be a continuous scalar function of the vector argument z, and let $V(0) = 0$. Consider the first difference

$$\Delta V = V(Az[n] + B\xi[n]) - V(z[n])$$

of the function $V(z)$ along the system (S, φ). Suppose that the inequality

$$\Delta V \leq -W$$

holds, where $W(z[n], \xi[n])$ $(n = 0, 1, 2, ...)$ is a positive definite quadratic form of the vectors $z[n]$ and $\xi[n]$.

(I). If $V(z[n]) \geq 0$ $(n = 0, 1, ...)$ for some solution $z[n]$ of the system (S, φ), then

$$|z[n]|, \ |\xi[n]| \in l_2(0, \infty)$$

and, consequently,

$$|z[n]| \to 0, \quad |\xi[n]| \to 0 \quad \text{as } n \to \infty.$$

(II). If $z[n] \to 0$ as $n \to \infty$, then $V(z[n]) > 0$ for $z[n] \neq 0$ $(n = 0, 1, ...)$.

(III). If $z[n] \neq 0$ as $n \to \infty$, then $V(z[n]) \to -\infty$ as $n \to \infty$.

Lemma 2. (generalized Kalman-Szegő-Popov lemma). Suppose constant matrices A, B of dimensions $\nu \times \nu$ and $\nu \times \rho$, respectively, are given, and a Hermitian form

$$\Omega(z, \xi, H) \equiv z^* Hz - (Az + B\xi)^* H(Az + B\xi) + U(z, \xi),$$

where the $\nu \times \nu$ matrix $H = H^*$ is to be found, the form $U(z, \xi)$ is Hermitian, and the vectors

z and ζ have, respectively, orders ν and p, the rank of the $\nu \times p\nu$-matrix $\|B, AB, ..., A^{\nu-1}B\|$ is equal to ν. Then, for the existence of a matrix $H = H^*$ such that the form $\Omega(z, \zeta, H)$ is a positive definite form of the vectors z and ζ, it is necessary and sufficient that for all λ, $|\lambda| = 1$, such that $\det(A - \lambda I) \neq 0$, the form $U(-(A - \lambda I)^{-1}B\zeta, \zeta)$ be a positive definite form of the vector argument ζ.

Lemma 3. Consider the system

$$z[n+1] = f(n, z[n]),$$

where the ν-vector function $f(n, z)$ is bounded when the arguments lie in a bounded domain. Suppose that there exists a function $V(z)$, continuous in z, satisfying the conditions: 1) $V(z) \to \infty$ as $|z| \to \infty$; 2) there exist $\xi_0 > 0$ and a continuous function $\alpha(z) > 0$, defined for $|z| \geq \xi_0$, such that, for any solution

$z[n]$ of the system under consideration, when $|z[n]| \geq \xi_0$ one has $\Delta V \leq -\alpha(z[n])$. Choose a number $\eta > 0$ so that $F = E\{V(z) \leq \eta\} \supset E\{|z| \leq \xi_0\}$.

Then, for solutions $z[n]$ of the system under consideration, the following hold: a) from $z[n_0] \in F$ it follows that $z[n] \in F$ for $n \geq n_0$; b) every solution $z[n]$, starting from some instant n_0, enters F.

Lemmas 1 and 3 are proved analogously to Lemmas 1 (^{1}) and 1 (^{2}). The simple proof of Lemma 2, different from the proof in (^{3}), consists in reducing it to Lemma 2 (^{1}). Setting

$$P = (A + \rho I)^{-1}(A - \rho I), \qquad Q = (I - P)B,$$

$$z = \frac{1}{\rho}(I - P)\left(x - \frac{1}{2}B\zeta\right),$$

$$\xi = {}^1/_2\zeta, \qquad \lambda = \rho(1 + i\omega)/(1 - i\omega),$$

where $-\infty \leq \omega \leq +\infty$, and the number ρ, $|\rho| = 1$, is such that $\det(A + \rho I) \neq 0$, we obtain

$$(Az + B\zeta)^* H(Az + B\zeta) - z^* Hz = 2 \operatorname{Re} x^* H(Px + Q\xi).$$

It is easy to verify that all the conditions of Lemma 2 are thereby transformed into the corresponding conditions of Lemma 2 (^{1}), from which the assertion of the lemma being proved follows. This device was used by Yu. A. Dmitriev in (^{4}), where an analogous lemma was formulated for a Hermitian form $U(z,\zeta)$ of a vector variable z and a scalar variable ζ of a special kind, under the assumption that the eigenvalues of the matrix A lie inside the unit circle, except perhaps for a single eigenvalue on the unit circumference.

4°. **Proof of Theorem 1.** (I). Construct the function $V(z) = z^* Hz$, where H is a Hermitian matrix satisfying the conditions of Lemma 1. We have $\Delta V = -\Omega(z,\zeta,H) - F(\zeta,\sigma)$, where Ω has the form indicated in Lemma 2, and $U(z,\zeta) = -F(\zeta,\sigma)$ with σ from (S). From condition $\{F\}$ the condition of Lemma 2 follows. Therefore there exists a matrix $H = H^*$ such that $\Omega(z,\zeta,H)$ is a positive definite form of z and ζ. From condition $\{\tilde{F}\}$ it follows that $\Delta V \leq -\Omega(z,\zeta,H)$. If $z[n] \to 0$ as $n \to \infty$, then, by Lemma 1, $V(z[n]) > 0$ $(n = 0,1,\ldots)$. Let the number $\varepsilon > 0$ be such that $\Omega \geq (1 - e^{-2\varepsilon})V$. Then $\Delta V + (1 - e^{-2\varepsilon})V \leq 0$, whence $V(z[n]) \leq e^{-2\varepsilon n}V(z[0])$, $|z[n]| \leq ce^{-\varepsilon n}|z[0]|$ with some $c > 0$.

By Lemma 1, in addition, we have $|\zeta[n]| \in l_2(0,\infty)$, and, consequently, $\zeta[n] \to 0$ as $n \to \infty$. Thus, in this case, (A) holds.

Suppose now that $z[n] \not\to 0$ as $n \to \infty$. Then, by Lemma 1, $V(z[n]) \to -\infty$ as $n \to \infty$, i.e. the set $K\{V(z) < 0\}$ is nonempty. Let $z[0] \in K$; then $V(z[n]) < 0$ for $n = 0,1,2,\ldots$. Represent $V(z)$ in the form $V(z) = V_1(z) - V_2(z)$, where the quadratic forms $V_1(z)$ and $V_2(z)$ are nonnegative. Choose $\varepsilon > 0$ so that $\Omega \geq (e^{2\varepsilon} - 1)V_2$. Then $-\Delta V \geq \Omega \geq (e^{2\varepsilon} - 1)V_2 \geq (e^{2\varepsilon} - 1)(-V)$. Hence $-V(z[n]) \geq e^{2\varepsilon n}(-V(z[0]))$ and $|z[n]| \geq c e^{\varepsilon n}|z[0]|$. Consequently, case (B) holds.

(II). In the proof of point (I) only the conditions $\{F\}$ and $\{\tilde{F}\}$ were used, but not the concrete form of the nonlinearities φ, i.e. assertion (A) of Theorem 1 is valid.

Proof of Theorem 2. From the proof of Theorem 1 it follows that, for the system $(S_\alpha, \varphi_\alpha)$, there exists a function $V(z) = z^* H_\alpha z$ satisfying the conditions of Lemma 1. Let $\alpha_0 \in [0, 1]$. Then ΔV, in view of the system $(S_\alpha, \varphi_\alpha)$ for $\alpha \in \Delta(\alpha_0)$, where $\Delta(\alpha_0)$ is a sufficiently small open interval containing α_0, has the form

$$\Delta V = -\Omega_\alpha - F_\alpha,$$

where

$$\Omega_\alpha = -(A_\alpha z + B_\alpha \xi)^* H_\alpha (A_\alpha z + B_\alpha \xi) + z^* H_\alpha z - F_\alpha$$

is a positive definite quadratic form.

Then, by what has been proved, for all systems $(S_\alpha, \varphi_\alpha)$, $\alpha \in \Delta(\alpha_0)$, either case (A) or case (B) holds. Choosing from the system of $\Delta(\alpha)$, $\alpha \in [0, 1]$, a finite covering $\Delta_1, \dots, \Delta_k$, we obtain the assertion of Theorem 2.

Proof of Theorem 3. (I). Consider the function $V(z) = z^* H z$ with Hermitian matrix H. Suppose that, for the system (S, φ), the solution $z[n] \to 0$ as $n \to \infty$. From the proof of Theorem 1,

$$\Delta V \leq -\Omega(z, \xi, H),$$

where Ω is a positive definite form in z and ξ. Then, by Lemma 1, $V(z) > 0$ for $z \neq 0$. Apply this same function $V(z)$ to the investigation of the system (S, φ, f). We have

$$\Delta V = -\Omega(z, \xi, H) - F + f^* H f + 2 \operatorname{Re} f^* H (Az + B\xi).$$

Choose a number $\delta > 0$ such that $\Omega \geq \delta(|z|^2 + |\xi|^2)$. Since $\lim |f|/|z| = 0$ as $|z| \to \infty$, there exists a sufficiently large number ξ_0 such that, for $|z| \geq \xi_0$,

$$f^* H f + 2 \operatorname{Re} f^* H A z \leq \tfrac{1}{2}\delta |z|^2, \qquad 2 \operatorname{Re} f^* H B \xi \leq \tfrac{1}{4}\delta(|z|^2 - |\xi|^2).$$

Then, for $|z| \geq \xi_0$, it is true that

$$\Delta V \leq -\tfrac{1}{4}\delta |z|^2.$$

By Lemma 3 we are convinced of the validity of (I) of Theorem 3.

(II). Suppose that, for (S, φ), case (B) holds. By Lemma 1, $V(z[n]) \to -\infty$ as $n \to \infty$ for $z[n] \not\to 0$ as $n \to \infty$, i.e., the set $z^* H z < 0$ is nonempty. From the conditions $\{F\}$ and $\{\tilde{F}\}$, for sufficiently large ξ_0, when $|z| \geq \xi_0$ it will hold that

$$-\Delta V = \Omega + F - f^* H f - 2 \operatorname{Re} f^* H (Az + B\xi) \equiv W,$$

where the quadratic form W is positive definite, i.e.,

$$\Delta V \leq -W < 0.$$

Consequently, if

$$z[0] \in \{z^* H z < 0\}$$

and $|z| > \xi_0$, then

$$V(z[n]) < 0.$$

Let $V = V_1 - V_2$, where the quadratic forms V_1 and V_2 are nonnegative. Choose a number $\varepsilon > 0$ such that

$$W \geq (e^{2\varepsilon} - 1)V_2.$$

Then

$$-\Delta V \geq W \geq (e^{2\varepsilon} - 1)V_2 \geq (e^{2\varepsilon} - 1)(-V).$$

Hence

$$|z[n]| \geq \text{const} \cdot e^{\varepsilon n}.$$

Received
21 IV 1969

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Note: Figure translations are in progress. See original paper for figures.

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