ON THE APPROXIMATION OF EXTREMAL PROBLEMS

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ON THE APPROXIMATION OF EXTREMAL PROBLEMS

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In the present note we consider questions of constructing a sequence of extremal problems approximating the original extremal problem, both with respect to the optimal value of the functional and with respect to the set of elements realizing its value. In the latter case the idea of regularization of A. N. Tikhonov ([1,2])* is used.

Let E be a set of elements of arbitrary nature; Δ a given nonempty subset of the set E; $I(u)$ a functional defined on E. By the original extremal problem we shall mean the problem of finding

$$I^* = \inf_{u\in\Delta} I(u)$$

and an element $u^* \in \Delta$ for which $I(u^*) = I^*$, if such an element exists.

Suppose that for each natural number $n = 1, 2, ...$ there are given: a set E_n, its subset Δ_n, and a functional $I_n(u_n)$ on the set E_n.

Definition 1. We shall say that the sequence of extremal problems of minimizing the functional $I_n(u_n)$ on the set Δ_n, $n = 1, 2, ...$, approximates the original problem in the sense of the optimal value of the functional (“with respect to the functional”), if $\Delta_n \neq \emptyset$ ($\emptyset$ is the empty set), $n = 1, 2, ...$, and

$$I_n^* = \inf_{u_n\in\Delta_n} I_n(u_n) \to I^*$$

as $n \to +\infty$. We give necessary and sufficient conditions for approximation of the original problem “with respect to the functional.”

Let $\mathfrak{A}$ be a given set in some topological space satisfying the first axiom of countability, and let $\theta \in \overline{\mathfrak{A}}$. Suppose that for each $\alpha \in \mathfrak{A}$ nonempty subsets $\Delta^{+\alpha}$, $\Delta^{-\alpha}$, and $\widetilde{\Delta}^{\alpha}$ of E are given. For any $\alpha \in \mathfrak{A}$ put

$$I^*_{\Delta^{+\alpha}} = \inf_{u\in\Delta^{+\alpha}} I(u), \qquad I^*_{\Delta^{-\alpha}} = \inf_{u\in\Delta^{-\alpha}} I(u), \qquad I^*_{\widetilde{\Delta}^{\alpha}} = \inf_{u\in\widetilde{\Delta}^{\alpha}} I(u).$$

Definition 2. The original extremal problem is called Δ^+-stable (Δ^--stable) with respect to the family $\Delta^{+\alpha}$ $[\Delta^{-\alpha}]$, $\alpha \in \mathfrak{A}$, if

$$\underline{\lim}_{\alpha\to\theta} I^*_{\Delta^{+\alpha}} \geq I^* \quad \left(\overline{\lim}_{\alpha\to\theta} I^*_{\Delta^{-\alpha}} \leq I^*\right). \tag{1}$$

If the problem is simultaneously Δ^+- and Δ^--stable, then we shall call it Δ-stable (with respect to the families $\Delta^{+\alpha}, \Delta^{-\alpha}$, $\alpha \in \mathfrak{A}$).

Theorem 1. *The sequence of problems of minimizing the functional* $I_n(u_n)$ *on the set* Δ_n, $n = 1, 2, \ldots$, *approximates the original problem with respect to the functional if and only if, for every* $\alpha \in \mathfrak{A}$, *there exist nonempty subsets* $\Delta^{+\alpha}$, $\Delta^{-\alpha}$, *and* $\widetilde{\Delta}^{\alpha}$ *of the set* E, *for which the following conditions are fulfilled:*

A.1. *For each* $n = 1, 2, \ldots$ *there exist a constant* $\varepsilon_{n0} > 0$ *and a mapping* $P_n : E_n \to E$ *such that, for every sequence of elements* $u_n \in \Delta_n$ *satisfying the conditions* $I_n(u_n) - I_n^* < \varepsilon_{n0}$, $n = 1, 2, \ldots,$ the relations

$$P_n u_n \in \Delta^{+\alpha_n}, \quad \alpha_n \to \theta \quad \text{as } n \to +\infty, \qquad \overline{\lim}_{n\to+\infty}\{I(P_{nu}n) - I_n(u_n)\} \leqslant 0$$

hold.

A.2. There exists a constant $\varepsilon_0 > 0$, and for all $\alpha \in \mathfrak{A}$ and $n = 1, 2, \ldots$ mappings $Q_{n\alpha} : E \to E_n$ such that, for any $\alpha \in \mathfrak{A}$ and any $u \in \widetilde{\Delta}^{\alpha}$ for which $I(u) - I^*_{\widetilde{\Delta}^{\alpha}} < \varepsilon_0$, the relations

$$Q_{n\alpha} u \in \Delta_n, \quad n = 1, 2, \ldots, \qquad \overline{\lim}_{n\to+\infty}\{I_n(Q_{n\alpha}u) - I(u)\} \leqslant 0$$

hold.

A.3. For any $\alpha \in \mathfrak{A}$ there exists a sequence $\{\alpha_m\} \subset \mathfrak{A}$ such that

$$\overline{\lim}_{m\to+\infty} I^*_{\widetilde{\Delta}^{\alpha_m}} \leqslant I^*_{\Delta^{-\alpha}}.$$

A.4. The original extremal problem is Δ-stable with respect to the families $\Delta^{+\alpha}$, $\Delta^{-\alpha}$, $\alpha \in \mathfrak{A}$.

Let us now consider the question of the approximate construction of elements from the set

$$\Delta^* = \{u \in E;\ u \in \Delta,\ I(u) = I^* = \inf_{v\in\Delta} I(v)\}.$$

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* The principal results of the present note were reported at the "Lomonosov Readings" in April 1969 at the Faculty of Mechanics and Mathematics of Moscow State University.

We shall assume that on the set E some topology τ is given. Suppose that on E a functional $\Omega(u)$ is defined and, for each $n = 1, 2, ...,$ on the set E_n a functional $\Omega_n(u_n)$ is defined, by means of which the "regularized" functional

$$I_{n\chi_n}(u_n) = I_n(u_n) + \chi_n \Omega_n(u_n)$$

is constructed, as well as an element $v_n \in \Delta_n$ for which

$$I^*_{n\chi_n} = \inf I_{n\chi_n}(u'_n) \leqslant I_{n\chi_n}(v_n) < I^*_{n\chi_n} + \eta_n.$$

Here $\{\chi_n\}$ and $\{\eta_n\}$ are sequences of positive numbers converging to zero.

Lemma 1. Suppose that, in addition to conditions A.1–A.4, the following requirements are satisfied:

A.5. For any sequence of elements $u_n \in \Delta_n$, $n = 1, 2, ...,$ for which

$$\{I_n(u_n) - I^*_n\} \to +0 \quad \text{as } n \to +\infty,$$

the relation

$$\overline{\lim}_{n\to+\infty} \chi_n \Omega_n(u_n) \leqslant 0$$

holds.

A.6. For any sequence of elements $u_n \in \Delta_n$, $n = 1, 2, ...,$ for which

$$\{I_{n\chi_n}(u_n) - I^*_{n\chi_n}\} \to +0 \quad \text{as } n \to +\infty,$$

the relation

$$\underline{\lim}_{n\to+\infty} \chi_n \Omega_n(u_n) \geqslant 0$$

holds.

A.7. For all $n = 1, 2, ...$ the inequality

$$I_n(v_n) - I^*_n < \varepsilon_{n0},$$

holds, where ε_{n0} are the constants from condition A.1.

Then $I(P_{nv}n) \to I^*$ as $n \to +\infty$.

Lemma 2. Suppose that, in addition to conditions A.1–A.4, A.7, the following requirements are satisfied:

A.8. For any real $C > +\infty$, the set $\{u \in E : \Omega(u) \leqslant C\}$ is relatively τ-countably compact, i.e. every infinite subset of it has at least one τ-limit point.

A.9. For every sequence of elements $u_n \in \Delta_n$ for which

$$I_n(u_n) - I^*_n < \varepsilon_{n0}, \quad n = 1, 2, \dots,$$

the inequality

$$\overline{\lim}_{n\to+\infty} \Omega(P_{nu}n) \leqslant \overline{\lim}_{n\to+\infty} \Omega_n(u_n),$$

holds, where the constant $\varepsilon_{n0} > 0$ and the mapping P_n are from condition A.1.

A.10. There exists an $a_0 \in \mathfrak{A}$ such that the set $\widetilde{\Delta}^{\alpha_0} \cap \Delta^* \neq \emptyset$, and, for some $u^* \in \widetilde{\Delta}^{\alpha} \cap \Delta^{\alpha}$, the relations

$$I(u) - I^*_{\widetilde{\Delta}^{\alpha_0}} < \varepsilon_0, \qquad \overline{\lim_{n\to+\infty}}\, \Omega_n(Q_{na_0}(u^*)) \leq \Omega(u^*),$$

$$\lim_{n\to+\infty} \frac{I_n(Q_{na_0}(u^*)) - I_n^* + \eta_n}{\chi_n} = 0$$

hold.

Then the sequence $\{P_n v_n\}$ is relatively τ-countably compact.

Definition 3. The family $\Delta^{+\alpha}$, $\alpha \in \mathfrak{A}$, is called τ-Δ^+-correct if, for any sequence $a_n \to \theta$ as $n \to +\infty$, every τ-limit point of any sequence $\{u_n\}$, $u_n \in \Delta^{+\alpha_n}$, $n = 1, 2, ...$, is a limit point of the set Δ.

Theorem 2. Suppose that, in addition to A.1—A.10, the following conditions are fulfilled:

A.11. The functional $I(u)$ is lower semicontinuous on E.

A.12. The family $\Delta^{+\alpha}$, $\alpha \in \mathfrak{A}$, is τ-Δ^+-correct.

A.13. The set Δ is τ-closed.

Then every limit point of the sequence $\{P_n v_n\}$ (by Lemma 2 such points exist) belongs to the set Δ^*.

Theorem 3. Suppose that, in addition to A.1—A.13, the conditions are fulfilled:

A.14. The functional $\Omega(u)$ is lower semicontinuous on Δ^*.

A.15. The set $\widetilde{\Delta}^{\alpha_0} \cap \Delta^*_\Omega \neq \emptyset$, and the element u^* from condition A.10 belongs to the set $\widetilde{\Delta}^{\alpha_0} \cap \Delta^*_\Omega$. Here

$$\Delta^*_\Omega = \{u \in E,\ u \in \Delta^*,\ \Omega(u) = \inf_{v\in\Delta} \Omega(v)\}.$$

Then every limit point of the sequence $\{P_n v_n\}$ belongs to the set Δ^*_Ω.

Theorem 4. Suppose that, in addition to A.1—A.15, the condition is fulfilled:

A.16. The set Δ^*_Ω consists of a single element u^*, and $u^* \in \widetilde{\Delta}^{\alpha}$.

Then the entire sequence $\{P_n v_n\}$ τ-converges to u^*. Moreover,

$$\Omega(P_n v_n) \to \Omega(u^*)$$

as $n \to +\infty$.

Remark 1. From the last assertion there may follow convergence of the sequence $\{P_n v_n\}$ to u^* in a topology stronger than τ.

Lemma 2 and the subsequent theorems are easily extended to the case where the functional is computed not exactly, but with errors, provided the parameter χ_n is suitably coordinated with the magnitude of the errors.

Remark 2. If (instead of condition A.10), for an element $u^* \in \Delta^*$, there is a sequence of elements $u^{(m)} \in \widetilde{\Delta}^{\alpha_m}$, $\alpha_m \in \mathfrak{A}$, $m = 1, 2, ...,$

$$u^{(m)} \to u^*$$

as $m \to +\infty$, $\alpha_m \to \theta$ as $m \to +\infty$, then, under hypotheses analogous to those formulated above, one can prove convergence to u^* of some subsequence $\{P_{n_m} v_{n_m}\}$ of the sequence $\{P_n v_n\}$.

Remark 3. Approximating sequences of extremal problems can also be constructed under the assumption that only one of the two families $\Delta^{+\alpha}$, $\Delta^{-\alpha}$, $\alpha \in \mathfrak{A}$, is specified, if the corresponding family $\Delta_n^{+\alpha}$ or $\Delta_n^{-\alpha}$ is specified in E_n for each $n = 1, 2, ...$, and if one requires of the sequence of extremal problems on the minimum of the functional $I_n(u_n)$ on the set Δ_n a certain stability (with respect to the indicated families). We mean stability in the sense of the following definition.

Definition 4. The sequence of problems on the minimum of $I_n(u_n)$ on Δ_n is called Δ^+-stable (Δ^--stable) with respect to the family $\Delta_n^{+\alpha}$ ($\Delta_n^{-\alpha}$), $\alpha \in \mathfrak{A}$, $n = 1, 2, ...$, if for any sequence $\{a_n\} \subset \mathfrak{A}$, $a_n \to \theta$ as $n \to +\infty$, the inequality

$$\varliminf_{n\to+\infty} \{I^*_{n\Delta_n^{+a_n}} - I^*\} \geq 0 \quad \left[\varlimsup_{n\to+\infty} \{I^*_{n\Delta_n^{-a_n}} - I^*\} \leq 0\right]$$

is valid.

Here

$$I^*_{n\Delta_n^{\pm a_n}} = \inf_{u_n \in \Delta_n^{\pm a_n}} I_n(u_n).$$

In conclusion, let us briefly consider the Δ^+- and Δ^--stability of the original problem.

Definition 5. A family $\Delta^{-\alpha}$, $\alpha \in \mathfrak{A}$, is called τ-Δ^--correct if, for any sequence $\{\alpha_n\} \subset \mathfrak{A}$ converging to θ, any element $u \in \Delta$ is a τ-limit point for some sequence $\{u_m\}$, $u_m \in \Delta^{-\alpha_m}$, $m = 1, 2, ...$

Theorem 5. Suppose the following conditions are satisfied: 1) the functional $I(u)$ is τ-lower semicontinuous (upper semicontinuous) on the set Δ; 2) the family $\Delta^{+\alpha}$ ($\Delta^{-\alpha}$), $\alpha \in \mathfrak{A}$, is τ-Δ^+-correct (τ-Δ^--correct); 3) the set $\bigcup_{\alpha\in\mathfrak{A}} \Delta^{+\alpha}$ is relatively τ-countably compact, and the set Δ is τ-closed.

Then the original problem is Δ^+-stable (Δ^--stable), and condition 3) is needed only for Δ^+-stability.

Let an arbitrary set F, a topology σ on it, and a mapping $T : E \to F$ be given. We give sufficient conditions for the Δ^+- and Δ^--correctness of the families $\Delta^{+\alpha}$

and $\Delta^{-\alpha}$, $\alpha \in \mathfrak{A}$, in the topology $T^{-1}(\sigma)$ for the case when the sets Δ, $\Delta^{+\alpha}$, $\Delta^{-\alpha}$ have the form

$$\Delta = \{u \in E:\ T(u) \in \Gamma\}, \qquad \Delta^{\pm\alpha} = \{u \in E:\ T(u) \in \Gamma^{\pm\alpha}\},$$

where $\Gamma, \Gamma^{+\alpha}, \Gamma^{-\alpha}$, $\alpha \in \mathfrak{A}$, are given subsets.

Theorem 6. Suppose the family $\Gamma^{+\alpha}$ $(\Gamma^{-\alpha})$, $\alpha \in \mathfrak{A}$, is σ-Γ^{+}-correct (σ-Γ^{-}-correct), and $T\Delta = \Gamma$ for all $\alpha \in \mathfrak{A}$, $T\Delta^{-\alpha} = \Gamma^{-\alpha}$.

Then the family $\Delta^{+\alpha}$ $(\Delta^{-\alpha})$ is Δ^{+}- (Δ^{-})-correct (with respect to the topology $T^{-1}(\sigma)$).

Realizations of the general propositions given above in the case of optimal-control problems are contained in ([3], [4]), and will also be considered in a separate article.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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