ON A CLASS OF NONLINEAR OPERATOR EQUATIONS AND SOME EQUATIONS OF MECHANICS

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ON A CLASS OF NONLINEAR OPERATOR EQUATIONS AND SOME EQUATIONS OF MECHANICS

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In Sec. 1 of the present note we consider the question of solvability of operator equations of the form $F(x)+B(x)=\theta$ in a Banach space and prove for them the convergence of the Galerkin method. Here F is a monotone operator, and B is an orthogonal weakly continuous operator. For the case $B\equiv 0$, existence theorems were obtained in recent works ([12,14,15]). The first results on the justification of the convergence of the Galerkin method were obtained in ([1,2]).

The convergence of the Galerkin method for linear operator equations was investigated in ([4]), for nonlinear equations with completely continuous terms in ([3]), and with monotone operators in ([7]). Certain problems of mechanics lead to equations of the form $F(x)+B(x)=\theta$. In Sec. 2 of our note, with the aid of the theorems of Sec. 1, we prove the convergence of the Galerkin method for the nonlinear Navier–Stokes system of equations and for the nonlinear system of equations of large deflection of thin plates. Questions of convergence of the Galerkin method for the indicated equations of mechanics were previously considered in ([9,10]). In Sec. 3 we prove the convergence of the Galerkin method for boundary-value problems studied in ([14,16]). In the note, throughout, strong convergence is denoted by $\to$, weak convergence by $\rightharpoonup$.

1. Let V be a real Banach space, and V^* its conjugate; (z,x) is the value of the linear functional $z\in V^*$ at the element $x\in V$. Consider operators $F:V\to V^*$ and $B:V\to V^*$ satisfying the conditions: I. Monotonicity

$$(F(u)-F(v),u-v)\geqslant 0,\quad \forall u,v\in V.$$

II. Orthogonality

$$(B(u),u)=0,\quad \forall u\in V,\qquad B(\theta)=\theta.$$

III. Coercivity

$$(F(u),u)\geqslant \gamma(\|u\|)\|u\|,\quad \forall u\in V,$$

where γ is a real function of the nonnegative argument t,

$$\lim_{t\to+\infty}\gamma(t)=+\infty.$$

IV. F is hemicontinuous, and B is a weakly continuous operator (i.e., for any $v \in V$ the functionals

$$\varphi_v(u) = (F(u), v), \qquad \psi_v(u) = (B(u), v)$$

are, respectively, continuous and weakly continuous in u in V).

Theorem 1. *Let the space V be reflexive, and let the operators F and B, defined on all of V, satisfy conditions I–IV. Then, for any $z \in V^*$, the equation $F(u) + B(u) = z$ has at least one solution $u_0 \in V$.*

The main point of the proof consists in verifying the following fact. The sets

$$M_u = \{v : \ (F(u), u - v) + (B(v), u) \geqslant 0, \ v \in D_R\},$$

$$D_R = \{v : \ \|v\| \leqslant R, \ v \in V\}$$

form a centered system. Here the index u runs through all of V, and $R > 0$ is a sufficiently large number.

Let now V be a separable space and let the linear combinations of the elements $u^i \in V$ $(i = 1, \dots, n, \dots)$ be everywhere dense in V. Without loss of generality, one may assume that any finite number of the elements u^i is linearly independent. Consider a system of m equations with m real unknowns $a_1, \dots, a_m$

$$\left(F\left(\sum_{i=1}^{m} a_i u^i\right), u^j\right) + \left(B\left(\sum_{i=1}^{m} a_i u^i\right), u^j\right) = 0, \qquad j = 1, \dots, m. \quad (1)$$

Let $(a_{10}, \dots, a_{m0})$ be a solution of system (1). The element $a_0^m = \sum_{i=1}^{m} a_{i0} u^i$ is called the m-th Galerkin approximation to the solution of the equation $F(x) + B(x) = \theta$. Consider the following conditions. V. The operator F is strongly monotone, i.e. $(F(u) - F(v), u - v) \geq \gamma(\|u - v\|)\|u - v\|$, $\forall u, v \in V$, where γ is a continuous strictly increasing function of a nonnegative argument t, $\gamma(0) = 0$, $\lim_{t\to+\infty} \gamma(t) = +\infty$. VI. The operator F is bounded (i.e. maps bounded sets from V into bounded sets from V^*). VII. The equation $F(u) + B(u) = \theta$ has at most one solution in V.

Theorem 2. *Let V be a separable reflexive Banach space, and let the operators $F : V \to V^*$, $B : V \to V^*$, defined on all of V, satisfy conditions II, IV, V, VI, VII. Then the following assertions hold: a) the equation $F(u) + B(u) = \theta$ has a unique solution u_0 in V; b) for every natural m there exists at least one Galerkin approximation a_0^m to the solution of the equation $F(u) + B(u) = \theta$; c) $\lim_{m\to\infty} \|a_0^m - u_0\| = 0$ (here a_0^m is any Galerkin approximation for the given m).*

2. In this section applications to problems of mechanics will be considered. We shall prove only the convergence of the Galerkin method for these problems.

A. Let Ω be a bounded simply connected domain of n-dimensional Euclidean space with boundary S, and let $u = \{u_1, \dots, u_n\}$ be a vector function defined in Ω, $n \geq 2$. The integral $\int_\Omega \varphi \, d\Omega$ denotes n-dimensional Lebesgue integration, and derivatives are understood in the sense of S. L. Sobolev. We shall seek in the domain Ω a solution of the first boundary-value problem for the Navier—Stokes system of equations

$$-\nu \Delta u + \sum_{k=1}^{n} u_k \frac{\partial u}{\partial x_k} = -\operatorname{grad} p + f, \tag{2}$$

$$\operatorname{div} u = 0,$$

satisfying the condition

$$u|_S = 0. \tag{3}$$

Here $f = f(x) = \{f_1, \dots, f_n\}$, $x = \{x_1, \dots, x_n\}$, and p is a scalar function in Ω, $\operatorname{grad} p \in L_2$. Consider the space $V = \{u : u \in \dot{W}_2^1,\ \operatorname{div} u = 0\}$. By a generalized solution of the boundary-value problem (2), (3) we shall mean a function $u \in V$ satisfying, for all $\Phi \in V$, the identity

$$\int_\Omega \left[\nu \sum_{k=1}^{n} \frac{\partial u}{\partial x_k} \frac{\partial \Phi}{\partial x_k} - \sum_{k=1}^{n} u_k u \frac{\partial \Phi}{\partial x_k} \right] d\Omega = \int_\Omega f \Phi \, d\Omega.$$

Denoting the scalar product in V by $[\cdot, \cdot]$, we obtain

$$\int_\Omega \nu \sum_{k=1}^{n} \frac{\partial u}{\partial x_k} \frac{\partial \Phi}{\partial x_k} \, d\Omega = \nu [u, \Phi].$$

It is easy to show that $\int_\Omega \sum_{k=1}^{n} u_k u \frac{\partial \Phi}{\partial x_k} \, d\Omega$, for fixed $u \in V$, is a linear bounded functional of Φ. Denoting it by $B(u)$, we obtain

$$\int_\Omega \sum_{k=1}^{n} u_k u \frac{\partial \Phi}{\partial x_k} \, d\Omega = [B(u), \Phi].$$

Suppose

$$\int_\Omega f \Phi \, d\Omega = [\tilde{f}, \Phi], \quad \tilde{f} \in V, \quad \forall \Phi \in V.$$

To show that u_0 is a generalized solution, it is sufficient to show that u_0 is a solution of the equation

$$-\frac{1}{\nu}B(u) - \frac{1}{\nu}\tilde{f} = 0.$$

The Galerkin process for this equation gives the Galerkin process for problem (2), (3).

Theorem 3. Let ν, f be such that problem (2), (3) has a unique generalized solution. Then: a) for any natural m there exists at least one Galerkin approximation a_0^m to the solution of problem (2), (3); b) if u_0 is the solution of problem (2), (3), then

$$\lim_{m\to\infty} \|a_0^m - u_0\|_V = 0$$

(here a_0^m is any Galerkin approximation for the given m).

B. Let Ω be a bounded simply connected domain in the plane with boundary S. Consider the problem of determining the state of a clamped thin plate under large deflection. This problem reduces to finding the solution of a boundary-value problem for a system of nonlinear partial differential equations in the domain Ω

$$\frac{h^2 E}{12(1-\sigma^2)}\Delta^2\xi - \left(\frac{\partial^2\chi}{\partial y^2}\frac{\partial^2\xi}{\partial x^2} - 2\frac{\partial^2\chi}{\partial x\partial y}\frac{\partial^2\xi}{\partial x\partial y} + \frac{\partial^2\chi}{\partial x^2}\frac{\partial^2\xi}{\partial y^2}\right) = \frac{1}{h}P(x,y), \tag{4}$$

$$\frac{2}{E}\Delta^2\chi + 2\frac{\partial^2\zeta}{\partial x^2}\frac{\partial^2\xi}{\partial y^2} - 2\left(\frac{\partial^2\zeta}{\partial x\partial y}\right)^2 = 0$$

with boundary conditions

$$\zeta|_S = \chi|_S = \left.\frac{\partial\zeta}{\partial x}\right|_S = \left.\frac{\partial\zeta}{\partial y}\right|_S = \left.\frac{\partial\chi}{\partial x}\right|_S = \left.\frac{\partial\chi}{\partial y}\right|_S = 0. \tag{5}$$

Here h is the thickness of the plate; E is Young' s modulus; σ is Poisson' s ratio, $0 < \sigma < 1$; the unknown functions ζ are the deflection and χ is the stress function.

Consider the Hilbert space of vector-functions

$$V = \{u : u = \{\zeta, \chi\},\ \zeta, \chi \in \dot{W}_2^2\}$$

with scalar product

$$[u_1, u_2] = (\zeta_1, \zeta_2)_{\dot{W}_2^2} + (\chi_1, \chi_2)_{\dot{W}_2^2},$$

where $u_i = \{\zeta_i, \chi_i\}$, $i = 1, 2$,

$$(v, \varphi)_{\dot{W}_2^2} = \int_{\Omega} \sum_{|\alpha|=2} D^{\alpha} v D^{\alpha} \varphi \, dx \, dy.$$

The scalar product in $\mathcal{L}_2$ we denote by $(\cdot, \cdot)$. Let $v = \{v_1, v_2\} \in V$. For each fixed $u = \{\xi, \chi\}$, $u \in V$, the form

$$A(u, v) \equiv \frac{h^2 E}{12(1 - \sigma^2)} (\Delta \xi, \Delta v_1) + \frac{2}{E} (\Delta \chi, \Delta v_2)$$

defines a bounded linear functional on V. Denoting it by Au, we obtain

$$A(u, v) \equiv [Au, v],$$

where $A : V \to V$ is a bounded linear positive definite operator. For $u, v \in V$ consider the form

$$\begin{aligned} B(u, v) \equiv B(\xi, \chi, v_1, v_2) \equiv & \iint_{\Omega} \left[\frac{\partial^2 \chi}{\partial y^2} \frac{\partial \xi}{\partial x} \frac{\partial v_1}{\partial x} - \frac{\partial^2 \chi}{\partial x \partial y} \left(\frac{\partial \xi}{\partial x} \frac{\partial v_1}{\partial y} + \frac{\partial \xi}{\partial y} \frac{\partial v_1}{\partial x} \right) \right. \\ & \left. + \frac{\partial^2 \chi}{\partial x^2} \frac{\partial \xi}{\partial y} \frac{\partial v_1}{\partial y} \right] dx \, dy \\ & + 2 \iint_{\Omega} \left[\frac{\partial \xi}{\partial x} \frac{\partial^2 \xi}{\partial x \partial y} \frac{\partial v_2}{\partial y} + \frac{\partial \xi}{\partial y} \frac{\partial^2 \xi}{\partial x \partial y} \frac{\partial v_2}{\partial x} + \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial y} \frac{\partial^2 v_2}{\partial x \partial y} \right] dx \, dy. \end{aligned}$$

It can be proved that for fixed $u \in V$ the form $B(u, v)$ is a bounded linear functional on V (denote it by $B(u)$). We have

$$B(u, v) = [B(u), v].$$

Suppose that

$$\iint_{\Omega} \frac{1}{h} P v_1 \, dx \, dy = [\tilde{P}, v], \quad \tilde{P} \in V, \quad \forall v \in V.$$

By a generalized solution of problem (4), (5) we shall mean any element $u_0 \in V$ that is a solution of the equation

$$Au + B(u) = \tilde{P}.$$

The Galerkin approximations for this equation will also be Galerkin approximations to the solution of problem (4), (5).

Theorem 4. Let P be such that $\tilde{P}$ exists, and suppose problem (4), (5) has a unique generalized solution. Then: a) for any natu-

for natural m there exists at least one Galerkin approximation a_0^m for the solution of problem (4), (5); b) if u_0 is the solution of problem (4), (5), then

$$\lim_{s\to\infty} \|a_0^m - u_0\|_V = 0$$

(here a_0^m is any Galerkin approximation for the given m).

The proof of Theorems 3 and 4 uses lemmas from (11) on the orthogonality of operators of a special form and the theorems from Sec. 1 of the present work.

3. Let us consider the convergence of the Galerkin method for the boundary-value problem

$$A(u) \equiv \sum_{|\alpha|\le l} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u) = h(x) \quad (|\gamma| \le l),$$

$$D^\omega u|_S = 0 \quad \text{for } |\omega| \le l-1. \tag{6}$$

Here Ω is a bounded domain of n-dimensional space R^n with a piecewise smooth boundary S.

Suppose the following conditions are satisfied: VIII. For $|\alpha| \le l$, all A_α are continuous functions of their numerical arguments. There exists $p > 1$ such that for all u from the Sobolev space $W_p^l(\Omega)$, all $x \in \Omega$, and $|\alpha| \le l$, the inequalities

$$|A_\alpha(x, u(x), \dots, D^l u(x))| \le K \left\{ \sum_{|\beta|\le l} |D^\beta u(x)|^{p-1} + 1 \right\}, \quad K = \text{const} > 0$$

hold.

IX. In the Sobolev space $\dot{W}_p^l(\Omega)$ the inequality

$$\sum_{|\alpha|\le l} \langle A_\alpha(x, u, \dots, D^l u) - A_\alpha(x, v, \dots, D^l v),\, D^\alpha(u-v)\rangle \ge$$

$$\ge \gamma(\|u-v\|_{\dot{W}_p^l}) \|u-v\|_{W_p^l}$$

holds for all $u, v \in \dot{W}_p^l(\Omega)$, where $\gamma(t)$ is a real continuous increasing function of $t \ge 0$,

$$\gamma(0) = 0, \quad \lim_{t\to+\infty} \gamma(t) = +\infty, \quad \langle u, v\rangle = \int_\Omega uv\,dx.$$

In works (14, 16), using the Galerkin method it was shown that, for any $h \in W_{p'}^{-l}(\Omega)$, problem (6) has a unique solution $u_0 \in \dot{W}_p^l(\Omega)$, provided only that conditions VIII, IX are satisfied.

Theorem 5. Suppose conditions VIII, IX are satisfied. Then for any natural m there exists a unique Galerkin approximation $a_0^m \in \dot{W}_p^l(\Omega)$ for the solution of problem (6), and

$$\lim_{m\to\infty} \|a_0^m - u_0\|_{\dot{W}_p^l} = 0,$$

whatever the element $h \in W_{p'}^{-l}(\Omega)$ may be.

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Note: Figure translations are in progress. See original paper for figures.

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