ON THE SOLUTION OF PROBLEMS OF THE LINEAR THEORY OF VISCOELASTICITY OF CONTACT TYPE

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Abstract

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ON THE SOLUTION OF PROBLEMS OF THE LINEAR THEORY OF VISCOELASTICITY OF CONTACT TYPE

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Consider an isotropic linear viscoelastic medium in which the relation between the stress tensor σ_{ij} and the strain tensor ε_{ij} has the form ([1])

$$s_{ij} = \int_0^t R(t-\tau)\, de_{ij}(\tau); \qquad e_{ij} = \int_0^t \Pi(t-\tau)\, ds_{ij}(\tau);$$
$$\sigma = \int_0^t R_1(t-\tau)\, d[\theta - 3\alpha\vartheta]; \qquad \theta - 3\alpha\vartheta = \int_0^t \Pi_1(t-\tau)\, d\sigma(\tau). \tag{1}$$

Here $s_{ij} = \sigma_{ij} - \sigma\delta_{ij}$, $e_{ij} = \varepsilon_{ij} - {}^1/_3\theta\delta_{ij}$, $\sigma = {}^1/_3\sigma_{kk}$, $\theta = \varepsilon_{kk}$; ϑ is the temperature drop; α is the coefficient of thermal expansion.

We write relations (1) in operator form:

$$s^*_{ij} = R^* e^*_{ij}; \qquad e^*_{ij} = \Pi^* s^*_{ij};$$

$$\sigma^* = R^*_1[\theta^* - 3\alpha\vartheta^*]; \qquad \theta^* - 3\alpha\vartheta^* = \Pi^*_1\sigma^*. \tag{2}$$

On the basis of the "basic" operators R^*, Π^*, R^*_1, Π^*_1, one can construct others, for example:

$$\omega^* = {}^1/_3 R^*\Pi^*_1, \qquad \nu^* = (3R^*_1 - R^*)/(6R^*_1 + R^*) = (1-\omega^*)/(2+\omega^*).$$

We shall call the operator $R^*\Pi^* = 1$ the identity operator*. We arrive at the representation (2), in particular, if we apply to each function $f(t)$ the Laplace-Carson transform

$$f^*(p) = p\int_0^\infty e^{-pt} f(t)\, d, \tag{3}$$

where everywhere below we shall assume that the parameter p is real ([2]).

In view of the foregoing, the equilibrium equations of the quasistatic problem of viscoelasticity take the form

$$-\frac{2+\omega^*}{3\omega^*}\theta^*_{,i} + u^*_{i,jj} = -2\rho\pi^*_i + \frac{2}{3}\frac{\alpha}{\omega^*}\vartheta^*_{,i}. \tag{4}$$

If the properties of the material depend on temperature, then in relations (1) and (3) the true time t should be replaced by the reduced time t', according to the principle of temperature–time analogy,

$$t' = \int_0^t \frac{dt}{a_T(T)}.$$

Suppose that on each part $\Sigma^{(q)}$ of the boundary of the body Σ, occupying the volume V, boundary conditions of "contact" type are prescribed:

$$a^{(q)*}_{ik}\sigma^*_{kj}l_j + b^{(q)}_{ik}u^*_k = N^{(q)*}_i, \tag{5}$$

where $a^{(q)}_{ik}$, $b^{(q)}_{ik}$ are elements of matrices that may be sufficiently smooth functions of the coordinates on Σ ([3]), l_j are the direction cosines of the surface element, and $N^{(q)}_i$ are contact forces.

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* If $R(t,\tau)$ and $R_1(t,\tau)$ are not kernels of difference type, then the corresponding operators R^* and R^*_1 are called noncommutative.

The problem of the linear theory of viscoelasticity consists in integrating equations (4) subject to the boundary conditions (5). The solution of this problem is unique. Indeed, the solution of the corresponding elastic problem is unique in the case when $-1 \leq \nu < 1/2$ ([3]). If the relaxation kernels $R(t)$ and $R_1(t)$ are continuous monotonically decreasing functions, convex with respect to the abscissa axis and tending to a nonnegative number as $t \to \infty$, then the expressions

$$M(e_{ij}) \equiv \int_0^t s_{ij}(\xi)\, de^j_i(\xi)$$

and

$$M_1(\theta) \equiv \int_0^t \sigma(\xi)\, d[\theta(\xi) - 3\alpha\vartheta(\xi)]$$

are positive-definite ([4]), whence the uniqueness of the posed problem follows.

From dimensional considerations, the solution of the elastic problem (4), (5) can be represented in the form

$$u_i^* = \Pi^* \left[\varphi_1(\gamma^*, \omega^*) F_i^* + \sum_q \varphi_{2q}(\gamma^*, \omega) N_i^{(q)*} \right] + \alpha \varphi_3(\gamma^*, \omega^*) \vartheta_{,i}^*, \qquad (6)$$

where the dimensionless operator γ^* reflects the conditions of contact of the body with the external medium, on which the elements of the matrices $a_{ik}^{(q)}$ and $b_{ik}^{(q)}$ in the boundary conditions (5) may depend. If the external medium is an elastic body, then γ^* is proportional to ω^*.

If the material under consideration is such that ν^*, and hence also ω^*, are numbers, then the viscoelastic solution is obtained directly from (6). Otherwise, each function $\varphi^*(\omega^*)$ in (6) must be represented or approximated by some analytic expression in ω^*, for example, in the form of a series in positive and negative powers of ω^* ([2])

$$\varphi^*(\omega^*) = \sum_{i=-M}^{+N} a_i \omega^{*i} \approx b_0 + b_1 \omega^* + b_2 \frac{1}{\omega^*}. \qquad (7)$$

Theorem 1. *Each function $\varphi^*(\omega^*)$ in the solution (6) is representable in the form of a finite or infinite sum of the form*

$$\varphi^*(\omega^*) = \sum_{i=0}^{N} A_i g_{\beta_i}^*, \qquad g_{\beta_i}^* = \frac{1}{1 + \beta_i \omega^*}, \qquad (8)$$

where β_i are certain real nonnegative distinct numbers, $\beta_0 \equiv 0$.

Indeed, suppose that in the problem of viscoelasticity theory (4), (5) the quantity $\chi^* = (2 + \omega^*)/3\omega^*$ does not belong to the Cosserat spectrum ([5]). Construct a harmonic vector $\tilde{u}_i^*$ satisfying the boundary conditions (5), and set $u_i^* - \tilde{u}_i^* = w_i^*$. Now it is necessary to find a solution of the equation

$$\chi^* \omega_{j,ji}^* + w_{i,jj}^* = -\chi^* \tilde{u}_{j,ij}^* - 2\rho \Pi^* F_i^* + \frac{2}{3} \frac{\alpha}{\omega^*} \vartheta_{,i}^* \equiv f$$

with homogeneous boundary conditions (5)

$$a_{ik}^{(q)} \sigma_{kj}^* l_j + b_{ik}^{(q)} u_k^* = 0.$$

According to S. G. Mikhlin' s theorem ([5]), the solution of this problem can be represented in the form

$$w^* = \sum_{n=1}^{\infty} \left\{ \frac{\chi_n f_n}{\chi_n - \chi} u_n + \frac{f_n^{(1)}}{1+\chi} u_n^{(1)} + f_n^{(\infty)} u_n^{(\infty)} \right\},$$

where χ_n $(n = 1, 2, ...)$ denote the Cosserat eigenvalues distinct from -1 and $-\infty$, u_n denote the corresponding orthonormalized eigenfunctions; $\{u_n^{(\infty)}\}$, $\{u_n^{(1)}\}$ are complete systems of orthonormalized Cosserat functions corresponding to the eigenvalues $\chi = -\infty$ and $\chi = -1$; $f_n = (f, u_n)$, $f_n^{(1)} = (f, u_n^{(1)})$, $f_n^{(\infty)} = (f, u_n^{(\infty)})$. Consequently, the solution of the posed problem has the form (6), where each $\varphi^*(\omega^*)$

is represented in the form

$$\varphi^*(\omega^*) = \sum_{n=1}^{N} \frac{r + s\omega^*}{1 + \beta_n \omega^*} + \frac{q}{1 + 2\omega^*}, \tag{9}$$

$\beta_n = (1 - 3\chi_n)/2$ are real, positive, distinct numbers; r, s, q are certain functions of the coordinates. This gives the representation (8).

The functions $g_\beta(t)$ can be found either analytically from the solution of the Volterra equation of the second kind (1), or experimentally. Suppose, for example, that a thin-walled specimen 1 of thickness δ and length l_1 is twisted by a force $Q(t)$, which simultaneously stretches another specimen 2 of the same material, having working-part length l_2 and cross-sectional area F_2. If the displacement of the specimen is now prescribed in the form $u = \Pi_1(t)\dfrac{l_2}{F_2}$, and the dynamometer readings on the change in load $Q(t)$ are recorded, then the kernel $g_\beta(t)$ for $0 \le \beta \le 1/2$ will be found from the formula $g_\beta(t) = 1 - Q(t)$; $\beta = l_2 \left(\dfrac{6F_2}{F_1} l_1 + 2l_2 \right)$, $F_1 = \pi(b^2 - a^2)$, $b - a = \delta$. To determine the kernels $g_\beta(t)$ for other values of β, one may fasten specimen 2 from below, and apply the load $Q(t)$, which twists specimen 1, to the top of specimen 2, forcing the latter to work in compression. In this case $\beta = l_2/(2l_2 - \dfrac{6F_2}{F_1} l_1)$. If the material under investigation does not relax volumetrically, then instead of the thin-walled specimen 1 one may successively attach to specimen 2 a spring of prescribed stiffness (1).

The limits of variation of the numbers β_n for some cases may be established with the aid of the Cosserat spectrum. Let us first note that, by virtue of the uniqueness theorem for problem (4), (5), all the numbers β_n in expression (9) can vary only in the range $0 \le \beta_n < \infty$. If only displacements are prescribed on the boundary of the body, then the uniqueness theorem can be extended (6) to values $\chi \ge -1$. Therefore, in this case all the numbers β_n lie in the interval $2 \le \beta_n < \infty$. However, additional investigations show that in this case the numbers $\beta = 2$ and $\beta = 1/2$ may also occur in (9). For a sphere, if only displacements are prescribed on the boundary, $\beta_n = (7n+3)/2n$; if, however, only stresses are prescribed on the boundary (5), $\beta_n = n(n-1)/(2n^2+4n+3)$. Thus, the above may be summarized as follows.

Theorem 2. *The solution of problem (4), (5) of the linear theory of thermo-viscoelasticity exists and is unique. If the elastic solution of a certain problem is known, then the corresponding viscoelastic solution is found exactly.*

However, there exists only a small number of elastic problems in which the explicit dependence on Poisson' s ratio ν is known. Therefore it is of definite interest to seek the viscoelastic solution of a problem if, for several different values of Poisson' s ratio, either a numerical realization of the elastic solution is known, or it has been found experimentally, for example by an optical method of stress investigation.

Suppose that on the interval $(0,\infty)$ or on some subinterval (α,β) it is necessary to approximate the function $\varphi(\omega)$ by the following expression $P(\omega)$**:

$$P(\omega) = \frac{A(\omega)}{B(\omega)}; \qquad A(\omega) = \sum_{i=1}^{p} a_i\omega^i; \qquad B(\omega) = \sum_{i=1}^{q} b_i\omega^i, \tag{10}$$

where $B(\omega)$ has q real distinct roots. It is necessary to find

* Some examples of the solution of such problems are given in (1).
** To simplify the notation, we shall henceforth omit the asterisks on ω^* and φ^*.

the parameters a_i, b_j $(i=1,\dots,p;\ j=1,\dots,q)$ (which, generally speaking, depend on the coordinates) so that the deviation of $P(\omega)$ from $\varphi(\omega)$:

$$H_P = \max_{\alpha\le\omega\le\beta} |\varphi(\omega) - P(\omega)|$$

is minimal*.

Theorem 3. *Among all functions of class (10), the best approximation to the function $\varphi(\omega)$ is given by the function $P(\omega)$ for which all roots of the equation $B(\omega)=0$ are negative, while the parameters a_i and b_i at each point of the body are chosen, for prescribed values $\varphi_{(j)} \equiv \varphi(\omega_{(j)})$, from the solution of the following algebraic system of equations:*

$$\sum_{i=1}^{p} a_i\omega_{(j)}^i = \varphi_{(j)} \sum_{i=1}^{q} b_i\omega_{(j)}^i \qquad (b_0 \equiv 1;\ j=1,\dots,p+q-1). \tag{14}$$

Indeed, the function $\varphi(\omega)$ has the form (9). Let us reduce the expression $H(\omega) = \varphi(\omega) - A(\omega)/B(\omega)$ to a common denominator. The denominator of the resulting expression $H(\omega)$ does not vanish for any positive ω, while the numerator for positive ω has at least $p+q+1$ roots and is a continuous function (a polynomial of finite or infinite order). Therefore $H(\omega)$, in sufficient proximity to the roots of its numerator at certain points of the interval (α, β): $\omega_1 < \omega_2 < \dots < \omega_N$, will change sign at least $N \equiv p+q+2$ times: $\lambda_1, -\lambda_2, \dots, (-1)^{N-1}\lambda_N$ $(N = p+q+2)$. Consequently, by Chebyshev' s theorem (7), the function $P(\omega)$ constructed in this way will give the best approximation to the function $\varphi(\omega)$. In this case the estimate

$$H_Q \geq \max\{\lambda_1, \lambda_2, \dots, \lambda_V\}$$

holds.

Theorem 4. *In order that the chosen approximation (10) give an exact expression for the function $\varphi(\omega)$, it is necessary and sufficient that the values of the parameters b_i $(i = 1, \dots, q)$, determined from the system (11), be the same for all points of the body under consideration.***

The proof of this theorem follows from consideration of the determinants of the matrix of the algebraic system of equations (11).

Remark 1. For example, for approximating the expression $1/(2\nu - d)$, $d > 1$, on the interval $-1/2 \leq \nu \leq 1/2$, the best Chebyshev approximation by polynomials of degree n has the error (7)

$$(d - \sqrt{d^2 - 1})^n (d^2 - 1)^{-1},$$

whereas an approximation of the form

$$(a\omega + b)/(1 + \beta\omega)$$

with three prescribed values gives an exact expression (for $\beta = (2+d)2(d-1)^{-1} > 1/2$).

Remark 2. The approximation of the expression $\varphi(\omega)$ can also be sought in the form of a rational function with fixed poles β. For example,

$$\varphi(\omega) \approx A/(1 + 2\omega) + B/(1 + \tfrac{1}{2}\omega) + C/\omega + D.$$

Remark 3. If the contacting body is not elastic, the approximation may be chosen in the form of a rational function of two variables:

$$\varphi(\omega, \gamma) \approx (a + b\omega + c\gamma)/(1 + \beta\omega + d\gamma).$$

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* Note that approximation (7) is a special case of approximation (10).

** Or they would differ among the different points of the body by the magnitude of the error of the available elastic solution.

Note: Figure translations are in progress. See original paper for figures.