SOME THEOREMS ON BASES IN HILBERT AND BANACH SPACES

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Abstract

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SOME THEOREMS ON BASES IN HILBERT AND BANACH SPACES

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The purpose of the present note is to study sequences of coefficients in expansions of elements of uniformly convex or uniformly smooth (for the definition see, for example, ([5])) Banach spaces E with respect to arbitrary seminormalized* bases $\{e_k\}_1^\infty$ in E. The case of a Hilbert space is considered separately.

§ 1. **Theorem 1****. Let $\{e_k\}_1^\infty$ be a seminormalized basis in a Banach space E. Then:

a) If E is uniformly convex, then there exist numbers $A > 0$ and $r > 1$, depending on the basis $\{e_k\}_1^\infty$, such that in the expansion of any element $x \in E$,

$$x = \sum_1^\infty \alpha_k e_k,$$

the inequality holds

$$\|x\| \leq A \left(\sum_1^\infty |\alpha_k|^r \right)^{1/r}. \tag{1}$$

b) If E is uniformly smooth, then there exist numbers $B > 0$, $s < \infty$, depending on the basis $\{e^k\}_1^\infty$, such that in the expansion of any element $x \in E$,

$$x = \sum_1^\infty \alpha_k e_k,$$

the inequality holds

$$\|x\| \geq B \left(\sum_1^\infty |\alpha_k|^s \right)^{1/s}. \tag{2}$$

Corollary. If a Banach space E is simultaneously uniformly convex and uniformly smooth, then there exist numbers $r, s, A > 0, B > 0,\ 1 < r \leq s < \infty$, depending on the basis $\{e_k\}_1^\infty$, such that in the expansion of any element $x \in E$,

$$x = \sum_1^\infty \alpha_k e_k,$$

the inequality holds

$$B\left(\sum |\alpha_k|^s\right)^{1/s} \leq \|x\| \leq A\left(\sum_1^\infty |\alpha_k|^r\right)^{1/r}. \tag{3}$$

Let us apply Theorem 1 to the study of the question of stability of bases in uniformly smooth Banach spaces. We first introduce the following

Definition 1. A complete sequence $\{e_k\}_1^\infty$ in a Banach space E is called completely stable with respect to a positive sequence $\{\varepsilon_k\}_1^\infty$ if for any po-

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* A sequence $\{e_k\}_1^\infty \subset E$ is called a basis in E if every element $x \in E$ can be represented in a unique way in the form

$$x = \sum_1^\infty \alpha_k e_k.$$

The basis $\{e_k\}_1^\infty$ is called seminormalized if, for some $m > 0,\ M > 0$, the relation $m \leq \|e_k\| \leq M,\ k = 1, 2, ...,$ holds.

** This theorem was obtained jointly with V. I. Gurarii.

sequences $\{g_k\}_1^\infty$ satisfying the condition $\|g_k - e_k\| < \varepsilon_k,\ k = 1, 2, ...,$ the linear operator T, defined by the relation $Te_k = g_k,\ k = 1, 2, ...,$ can be represented in the form

$$T = I + S, \qquad \|S\| < 1.$$

Remark. Under the conditions of Definition 1, T is an isomorphism of E onto itself ([6]). Therefore, if $\{e_k\}_1^\infty$ is a basis, then $\{g_k\}_1^\infty$ is also a basis in E, i.e., from complete stability there follows the usual stability of the basis, as well as the stability of other linear-topological properties of the sequence.

Theorem 2. *Let $\{e_k\}_1^\infty$ be a quasi-normalized basis in a uniformly smooth Banach space E. Then there exist numbers $p > 1$ and $R > 0$, depending only on the basis $\{e_k\}_1^\infty$, such that for any positive sequence $\{\varepsilon_k\}_1^\infty$ satisfying the condition $\sum_1^\infty \varepsilon_k^p < R$, $\{e_k\}_1^\infty$ is completely stable relative to $\{e_k\}_1^\infty$.*

Proof. Choose the number p as conjugate, in the sense of Hölder, to s, where s is defined in (2), i.e. $p = s/(s-1)$. Let $\{g_k\}_1^\infty \subset E$ be such that $\|e_k - g_k\| \leq \varepsilon_k$, $k = 1, 2, \ldots$. Define on the linear span $L(\{e_k\}_1^\infty)$ of the sequence $\{e_k\}_1^\infty$ a linear operator T by the equalities $Te_k = g_k$, $k = 1, 2, \ldots$. Let $T - I = S$; then S is also a linear operator, for the moment defined on L. Estimate $\|S\|$, putting $x = \sum_1^n \alpha_k e_k$, and applying (2) and Hölder' s inequality:

$$\|Sx\| = \|(T-I)x\| = \|Tx - x\| = \left\| \sum_1^n \alpha_k (g_k - e_k) \right\| \leq$$

$$\leq \sum_1^n |\alpha_k| \varepsilon_k \leq \left(\sum_1^n |\alpha_k|^s \right)^{1/s} \left(\sum_1^n \varepsilon_k^p \right)^{1/p} \leq \frac{\|x\|}{B} R^{1/p}.$$

We shall assume R chosen so that $R^{1/p}/B < 1$. Then $\|S\| < 1$, and consequently the operator T is an isomorphism of $L(\{e_k\}_1^\infty)$ onto $L(\{g_k\}_1^\infty)$. Extending the bounded operator T to all of E by continuity, we obtain a linear operator $\widetilde{T}$, which is an isomorphism of E onto itself and such that

$$\widetilde{T} e_k = g_k, \qquad k = 1, 2, \ldots, \qquad \widetilde{T} = I + S, \qquad \|S\| < 1.$$

The theorem is proved.

As far as we know, Theorem 2 is new also for the special case of Hilbert space, improving the condition $\sum_1^\infty \varepsilon_k < R$ in the theorem of Krein-Milman-Rutman ([4]).

§ 2. The starting point of the results of this section is inequality (3). A basis $\{e_k\}_1^\infty$ for which this inequality holds will be called an $\{r, s\}$-basis. By the pair of numbers $\{r, s\}$ one can classify bases in E. Thus, the class of all quasi-normalized unconditional* bases in Hilbert space H, by the theorem of Gelfand ([3]), coincides with the class of all $\{2, 2\}$-bases.

The exact least upper (greatest lower) bound of the numbers r (respectively s) for which inequality (3) holds will be called the lower (upper) degree of the basis $\{e_k\}_1^\infty$ and denoted respectively by $\rho = \rho(\{e_k\}_1^\infty)$, $\sigma = \sigma(\{e_k\}_1^\infty)$. If the least upper or greatest lower bound is not attained, then we shall use parentheses, applying the term (ρ, σ)-basis, and if

* A basis $\{e_k\}_1^\infty$ is called unconditional if it remains a basis under any permutation of its elements. A basis that is not unconditional is called conditional.

is attained—square brackets. Different brackets, for example $[\rho, \sigma)$ or $(\rho, \sigma]$, are understood in the natural way. If $\rho(\{e_k\}_1^\infty) \leq r$, $\sigma(\{e_k\}_1^\infty) \geq s$, $1 < r \leq s < \infty$, then we shall call $\{e_k\}_1^\infty$ an $\langle r, s \rangle$-basis.

The main result of this section is the non-improvability of inequality (3) for the case of a separable Hilbert space H, in the sense that for arbitrary numbers r and s, $1 < r \leq s < \infty$, there exists in H an $\langle r, s\rangle$-basis. An essential role in this is played by the systems

$$\{|t|^{\alpha} \cos nt\}_{-\infty}^{\infty}, \qquad \{|t|^{-\alpha} \cos nt\}_{-\infty}^{\infty}, \qquad 0 < \alpha < {}^1\!/_2, \tag{4}$$

which, as follows from the results of K. I. Babenko ([2]), are conditional bases in the closure of their linear span in $L_2[-\pi, \pi]$. Along the way, a number of theorems are obtained on the coefficients of expansions with respect to bases in arbitrary Banach spaces.

Theorem 3. Let $\{e_k\}_1^{\infty}$ be a quasinormalized sequence in a Banach space E satisfying the conditions:

1. $$\left\|\sum_1^n e_k\right\| \geq Kn^r \qquad (K > 0,\ r > 0), \qquad n = 1, 2, \dots .$$

2. For any finite sets of numbers $\{\alpha_i\}_1^n,\ \{\beta_i\}_1^n$, from the condition $0 \leq \alpha_k \leq \beta_k,\ k = 1, 2, \dots, n$, it follows that
$$\left\|\sum_1^n \alpha_k e_k\right\| \leq \left\|\sum_1^n \beta_k e_k\right\|.$$

If $\{\alpha_k\}_1^{\infty}$ is a positive monotonically decreasing sequence such that $\{\alpha_k\}_1^{\infty} \in l_p$, where $p > 1/r$, then the series $\sum_1^{\infty} \alpha_k e_k$ diverges.

Theorem 4. If a quasinormalized sequence $\{e_k\}_1^{\infty}$ in a Banach space E, for some r and K, $0 < r \leq 1,\ 0 < K < \infty$, satisfies the condition

$$\left\|\sum_1^n e_k\right\| \leq Kn^r, \qquad n = 1, 2, \dots,$$

and the positive sequence $\{\alpha_k\}_1^{\infty} \downarrow 0$ belongs to l_p, where $0 < p < 1/r$, then the series

$$\sum_1^{\infty} \alpha_k e_k$$

converges.

With the aid of several lemmas it is verified that the bases (4) satisfy the conditions of Theorems 3 and 4 (with the corresponding choice of the parameter α). Using these considerations, the following Theorems 5–7 are established.

Theorem 5. Let $\{\alpha_k\}_1^\infty$ be a numerical sequence such that $|\alpha_1| \geq |\alpha_2| \geq \cdots$. In order that, for every quasinormalized basis $\{e_k\}_1^\infty$ in H, the series

$$\sum_1^\infty \alpha_k e_k$$

converge, it is necessary and sufficient that $\{\alpha_k\}_1^\infty \in l_p$ for every $p > 1$.

Theorem 6. Let $\{\alpha_k\}_1^\infty$ be a numerical sequence such that $|\alpha_1| \geq |\alpha_2| \geq \cdots$. In order that in H there exist a quasinormalized basis $\{e_k\}_1^\infty$ such that the series

$$\sum_1^\infty \alpha_k e_k$$

converges, it is necessary and sufficient that $\{\alpha_k\}_1^\infty \in l_p$ for some $p,\ 1 < p < \infty$.

Theorem 7. For arbitrary numbers $r, s,\ 1 < r \leq s < \infty$, in a separable Hilbert space H there exists an $\langle r, s\rangle$-basis.

Theorem 7 generalizes M. Sh. Altman' s result on the existence, in a separable Hilbert space, of a basis that is neither Hilbertian nor Besselian ([1]). As an application of Theorem 7 we obtain

a negative solution to the question of the existence in a Hilbert space H of a universal basis $\{e_k\}_1^\infty$, i.e., one such that every normalized basis $\{g_i\}_1^\infty$ in H is equivalent* to some subsequence $\{e_{k_i}\}_1^\infty$ of the basis $\{e_k\}_1^\infty$.

Theorem 8. *In a separable Hilbert space H there does not exist a universal basis.*

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* Two bases $\{e_k\}_1^\infty$ and $\{g_k\}_1^\infty$, respectively in Banach spaces E and G, are called equivalent if there exists an isomorphism T of E onto G such that $Te_k = g_k,\ k = 1, 2, ...$

Note: Figure translations are in progress. See original paper for figures.