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1970

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Abstract

Full Text

UDC 517.948:513.88

I. M. KOVALCHIK

LINEAR EQUATIONS WITH FUNCTIONAL DERIVATIVES

(Presented by Academician N. N. Bogolyubov, 17 III 1970)

1. Let X be a normed space, \mathbf{R} the real line, and $J(x)$ a functional on X . The Fréchet derivative and the functional derivative of order k ($k = 1, \dots, n$) of the functional $J(x)$ (for the definition see, for example, ⁽¹⁻³⁾) will be denoted respectively by the symbols $J_{x\dots x}^{(k)}$ and $\delta^k J(x)/\delta x(t_1) \dots \delta x(t_k)$.

Consider the Cauchy problem

$$J_{x\dots x}^{(n)} + P_1(x)J_{x\dots x}^{(n-1)} + \dots + P_{n-1}(x)J'_x + P_n(x)J = Q(x), \quad (1)$$

$$J_{x\dots x}^{(k)} \Big|_{x=x_0} = J_0^k \quad (k = 0, 1, \dots, n-1), \quad (2)$$

where $P_k(x)$ ($k = 1, \dots, n-1$), for each fixed value of x , are linear operators from the space $\mathcal{L}_{n-k}(X, \mathbf{R})$ (continuous $n-k$ -linear mappings $X \times \dots \times X$ into \mathbf{R}) into the space $\mathcal{L}_n(X, \mathbf{R})$, while $P_n(x)$ and $Q(x)$ are elements of the space $\mathcal{L}_n(X, \mathbf{R})$ ⁽⁴⁾.

If, as the space X , one takes a space of functions and in a special way chooses the operators $P_k(x)$ in equation (1), then a particular case of this equation, for $n = 2$, will be, for example, the equation

$$\frac{\delta^2 J(x)}{\delta x(t_1)\delta x(t_2)} + p(x, t_1, t_2) \frac{\delta J(x)}{\delta x(t_1)} + q(x, t_1, t_2)J(x) = f(x, t_1, t_2). \quad (3)$$

We shall call equation (1) completely solvable if the initial conditions (2) uniquely determine a solution in some neighborhood of the point x_0 .

We shall call a general solution of equation (1) a functional $x \rightarrow \Phi(x, C_1, \dots, C_n)$, where C_1, \dots, C_n are arbitrary constants, which satisfies equation (1) for all values of these constants and, for prescribed initial conditions (2), the arbitrary constants can be chosen so that the given functional satisfies conditions (2).

The conditions for complete solvability of problem (1), (2) follow from the Frobenius theorem ⁽⁴⁾. In particular, for the solvability of equation (3) (the

case $n = 2$, taken for simplicity) it is necessary and sufficient that for any $t_1, t_2, t_3 \in [a, b] \subset \mathbf{R}$ the following conditions hold:

1°.

$$q(x, t_1, t_2) = q(x, t_2, t_1).$$

2°.

$$f(x, t_1, t_2) = f(x, t_2, t_1).$$

3°.

$$p(x_0, t_1, t_2)J_0^1(t_1) = p(x_0, t_2, t_1)J_0^1(t_2), \quad \text{where} \quad \left. \frac{\delta J(x)}{\delta x(t)} \right|_{x=x_0} = J_0^1(t).$$

4°.

$$\frac{\delta q(x, t_1, t_2)}{\delta x(t_3)} - p(x, t_1, t_2)q(x, t_1, t_3) = \frac{\delta q(x, t_1, t_3)}{\delta x(t_2)} - p(x, t_1, t_3)q(x, t_1, t_2).$$

5°.

$$\frac{\delta f(x, t_1, t_2)}{\delta x(t_3)} - p(x, t_1, t_2)f(x, t_1, t_3) = \frac{\delta f(x, t_1, t_3)}{\delta x(t_2)} - p(x, t_1, t_3)f(x, t_1, t_2).$$

6°.

$$\frac{\delta p(x, t_1, t_2)}{\delta x(t_3)} = \frac{\delta p(x, t_1, t_3)}{\delta x(t_2)}.$$

7°.

$$\left[\frac{\delta p(x, t_1, t_2)}{\delta x(t_3)} - p(x, t_1, t_2)p(x, t_1, t_3) \right] p(x, t_2, t_1) = \left[\frac{\delta p(x, t_2, t_3)}{\delta x(t_1)} - p(x, t_2, t_1)p(x, t_2, t_3) \right] p(x, t_1, t_2).$$

The conditions written down are an analogue of the well-known conditions of complete solvability for the Pfaff equation in the theory of ordinary differential equations. The corresponding results for equations containing the derivative of a mapping of one finite-dimensional space into another finite-dimensional space belong to A. I. Perov ⁽⁵⁾.

2. As usual, we shall call a linearly independent system of n solutions a fundamental system of solutions.

Theorem 1. Let $J_1(x), \dots, J_n(x)$ be a fundamental system of solutions of the completely solvable equation (1) for $Q = 0$. Then the general solution of this equation has the form

$$J(x) = C_1 J_1(x) + \dots + C_n J_n(x),$$

where C_1, \dots, C_n are arbitrary constants.

Consider the equation

$$\begin{aligned}
 L(J) \equiv & p_0(x)\delta^n J(x)/\delta x(t_1) \dots \delta x(t_n) + p_1(x, t_n)\delta^{n-1} J(x)/\delta x(t_1) \dots \\
 & \dots \delta x(t_{n-1}) + \dots + p_{n-1}(x, t_2, \dots, t_n)\delta J(x)/\delta x(t_1) + \\
 & + p_n(x, t_1, \dots, t_n)J(x) = q(x, t_1, \dots, t_n).
 \end{aligned} \tag{4}$$

Theorem 2. Suppose that equation (4) is completely solvable. Let the functional $K(x, y)$, for each fixed value of y , satisfy, with respect to x , the equation

$$L(J) = 0$$

and the conditions

$$K(x, y)|_{y=x} = 0,$$

$$\delta^k K(x, y)/\delta x(t_1) \dots \delta x(t_k)|_{y=x} = 0 \quad (k = 1, \dots, n-2),$$

$$\delta^{n-1} K(x, y)/\delta x(t_1) \dots \delta x(t_{n-1})|_{y=x} = 1.$$

Then the solution of equation (4) under the initial conditions

$$J(x)|_{x=x_0} = 0, \quad \delta^k J(x)/\delta x(t_1) \dots \delta x(t_k)|_{x=x_0} = 0 \quad (k = 1, \dots, n-1)$$

has the form

$$\begin{aligned}
 J(x) = & \left\{ \int_a^b [x(t) - x_0(t)] dt \right\}^{1-n} \int_0^1 K[x, s(x - x_0) + x_0] ds \times \\
 & \times \int_a^b \dots \int_a^b q[s(x - x_0) + x_0, t_1, \dots, t_n] \prod_{j=1}^n [x(t_j) - x_0(t_j)] dt_j.
 \end{aligned}$$

Let us note that there exists a class of equations with functional derivatives (in particular, equations for which the functional derivatives of a fundamental system of solutions up to order $n - 1$ do not depend on the points at which they are evaluated) such that for it one can introduce the notion of the Wronski determinant and establish a number of results connected with this determinant, for example, the Ostrogradskii-Liouville formula.

3. Considering the completely solvable equation (4) under the initial conditions

$$J(x_0) = J_0, \quad \delta^k J(x) / \delta x(t_1) \dots \delta x(t_k) \Big|_{x=x_0} = J_0^k \quad (k = 1, \dots, n-1), \quad (5)$$

we obtain its solution in the form

$$\begin{aligned} J(x) = & \varphi\left(1, x, x_0, J_0, \int_a^b J_0^1(t_1)[x(t_1) - x_0(t_1)] dt_1, \dots, \right. \\ & \int_a^b \dots \int_a^b J_0^{n-1}(t_1, \dots, t_{n-1}) \times \\ & \left. \times \prod_{j=1}^{n-1} [x(t_j) - x_0(t_j)] dt_j\right). \end{aligned} \quad (6)$$

where $\varphi(s, x, x_0, z_0, z_0', \dots, z_0^{(n-1)})$, for fixed x and x_0 , is the solution of the problem

$$\begin{aligned} a_0(s) d^n z / ds^n + a_1(s) d^{n-1} z / ds^{n-1} + \dots + a_n(s) z &= b(s) \quad (0 \leq s \leq 1), \\ z(0) = z_0, \quad d^{kz} / ds^k \Big|_{s=0} = z_0^{(k)} \quad (k = 1, \dots, n-1), \\ a_0(s) &= p_0[s(x - x_0) + x_0], \\ a_k(s) &= \int_a^b \dots \int_a^b p_k[s(x - x_0) + x_0, t_1, \dots, t_k] \prod_{j=1}^k [x(t_j) - x_0(t_j)] dt_j, \\ b(s) &= \int_a^b \dots \int_a^b q[s(x - x_0) + x_0, t_1, \dots, t_n] \prod_{j=1}^n [x(t_j) - x_0(t_j)] dt_j. \end{aligned}$$

Here the functional derivatives may be generalized functions of the parameters t_1, \dots, t_n .

Consequences of formula (6) are the results on the solution of concrete linear differential equations with functional derivatives obtained by a number of authors (6-8).

4. Let us take the completely solvable equation

$$\begin{aligned} \delta^n J(x) / \delta x(t_1) \dots \delta x(t_n) + a_1 p(t_n) \delta^{n-1} J(x) / \delta x(t_1) \dots \delta x(t_{n-1}) + \\ + a_2 p(t_{n-1}) p(t_n) \delta^{n-2} J(x) / \delta x(t_1) \dots \delta x(t_{n-2}) + \dots \\ \dots + a_{n-1} p(t_2) \dots p(t_n) \delta J(x) / \delta x(t_1) + a_{np}(t_1) \dots p(t_n) J(x) = 0 \end{aligned} \quad (7)$$

and for it form the characteristic equation

$$\lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n = 0.$$

Let the root λ_i of the characteristic equation occur m_i times, where

$$i = 1, \dots, p, \quad m_i \geq 1, \quad \sum_{i=1}^p m_i = n.$$

Theorem 3. The functionals

$$J_{ij}(x) = \left[\int_a^b p(t)x(t) dt \right]^{m_i-j} \exp \left[\lambda_i \int_a^b p(t)x(t) dt \right]$$

$$(i = 1, \dots, p; \quad j = 1, \dots, m_i)$$

form a fundamental system of solutions of equation (7).

5. As with ordinary differential equations, an equation in functional derivatives of the second order can be reduced to an equation resembling the Riccati equation.

We shall call the functional analogue of the Riccati equation an equation of the form

$$\delta J(x)/\delta x(t) = P(x, t)J^2(x) + Q(x, t)J(x) + R(x, t). \quad (8)$$

More general, in comparison with equation (8), is the equation

$$\delta J(x)/\delta x(t) = f(J, x, t), \quad (9)$$

which was already investigated by P. Lévy ⁽⁹⁾.

The necessary and sufficient condition for solvability of equation (9) has the form

$$\begin{aligned} & \delta f(J, x, t_1)/\delta x(t_2) + [\partial f(J, x, t_1)/\partial J] f(J, x, t_2) \\ & = \delta f(J, x, t_2)/\delta x(t_1) + [\partial f(J, x, t_2)/\partial J] f(J, x, t_1) \end{aligned} \quad (10)$$

for arbitrary $t_1, t_2 \in [a, b]$.

It follows immediately from (10) that there is a solvability condition for equation (8). For equation (8) there are theorems analogous to the theorems for the ordinary Riccati equation. Let us give, for example, the following result.

Theorem 4. *If the coefficients $P(x, t)$, $Q(x, t)$ and the derivative $\delta P(x, t_1)/\delta x(t)$ do not depend on t , then equation (8) can be reduced to the form*

$$\delta J_1(x)/\delta x(t) = J_1^2(x) + R_1(x, t). \quad (8')$$

In this case equation (8') will be completely solvable if

$$R_1(x, t) = \varphi \left[\int_a^b x(t) dt \right],$$

where $\varphi(u)$ is an ordinary differentiable function.

The actual solution of equation (8) or (8') follows immediately from the solution of the corresponding ordinary differential equation; moreover, the required result also holds for the general equation (9) (cf. (10)).

Theorem 5. *If equation (9) is completely solvable, then in some open ball there exists a unique solution of equation (9) satisfying the condition $J(x_0) = J_0$. This solution has the form $\varphi(1, x, x_0, J_0)$, where $\varphi(s, x, x_0, z_0)$, for fixed x and x_0 , is the solution of the problem*

$$dz/ds = b(s, z, x, x_0) \quad (0 \leq s \leq 1), \quad z(0) = z_0,$$

where

$$b(s, z, x, x_0) = \int_a^b f[z, s(x - x_0) + x_0, t] [x(t) - x_0(t)] dt.$$

Let us note in conclusion that everything presented in this section can also be rewritten for an equation with a Fréchet derivative, and most of the results of the article are preserved for mappings of a linear topological space into \mathbf{R} .

I take this opportunity to express my sincere gratitude to Yu. L. Daletskii for useful discussions.

Lviv Polytechnic Institute

Received
14 III 1970

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