ON \(L(1,\chi)\) WITH A REAL DIRICHLET CHARACTER ON SPARSE SETS OF VALUES OF THE CHARACTER MODULUS

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Abstract

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ON $L(1,\chi)$ WITH A REAL DIRICHLET CHARACTER ON SPARSE SETS OF VALUES OF THE CHARACTER MODULUS

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The study of the functions

$$L(1,\chi_D)=\sum_{n=1}^{\infty}\chi(D,n)n^{-1},$$

where $\chi(D,n)$ denotes: $\chi_n(D)=\left(\frac{\pm D}{n}\right)$, the Jacobi symbol or zero; or $\chi_D(n)=\left(\frac{\pm D}{n}\right)$, the Kronecker symbol or zero, was carried out in ($^{1-8}$) and is connected with applications to the theory of quadratic forms and the theory of divisors of quadratic fields. Recently the problem of the distribution of values of $L(1,\chi_D)$ has acquired additional interest in connection with A. Selberg' s formula (^{9}) for the trace of Hecke operators in the space of modular forms.

In the present paper new information is given on the nature of the behavior of the functions $L(1,\chi_D)$, obtained on the basis of the method indicated in (^{10}) for estimating double sums with a quadratic Dirichlet character.

Notation: $D=dm+l,\ 1\le l\le d,\ (l,d)=1,\ m=1,2,\ldots;\ b_D$ is any sequence of zeros and ones; $\gamma=2$ or 1, according as $\chi(D,n)=\chi_n(D)$ or $\chi_D(n)$; $\tau_k(m)$ is the number of solutions of the equation $m=m_1\ldots m_k$ in integers $m_\nu\ge 1$,

$$H_k(N)=\sum_{m=1}^{N}\left[b_D\prod_{p|d}\left(1-\frac{\chi(D,p)}{p}\right)L(1,\chi_D)\right]^k,$$

where p is a prime number.

Main lemma. Uniformly with respect to N,d, and l we have

$$H_k(N)=\sum_{\substack{n=1\\(n,\gamma)=1}}^{N}\sum_{\substack{m=1\\(D,n)=1}}^{N}\frac{\tau_k(n^2)\chi(d^2,n)b_D}{n^2}+O\left(N^{9/10}d^{4/5}\ln^c dN\right)+$$

$$+O\left(\ln^c dN\left[\sum_{2\le n\le X}\sum_{\substack{r=1\\(r,n)=1}}^{n}\left|\sum_{\substack{m=1\\D\equiv r(\mathrm{mod}\,n)}}^{N} b_D-F\right|^2\right]^{1/2}\right),$$

where F is an arbitrary quantity not depending on l; $c=c(k)$; $X=N^{2/3}d^{3/10}$.

Thus the problem of the asymptotics for the moments of the function

$$b_D\prod_{p|d}\left(1-\frac{\chi(D,p)}{p}\right)L(1,\chi_D) \tag{1}$$

reduces only to the equidistribution of the sequence b_D "on average" over primitive arithmetic progressions. As is known, a broad class of sequences possesses a property of this kind. In particular, for $b_D=\mu^2(D)$, where μ is the Möbius function, the following is obtained.

Theorem 1. Uniformly with respect to N, d, and l we have

$$H_k(N)=N\sum_{\substack{n=1\\(n,\gamma)=1}}^{\infty}\sum_{r/n}\sum_{\delta=1}^{\infty}\frac{\tau_k(n^2)\chi(d',n)\mu(r)\mu(\delta)\varphi(r)}{(nr\delta)^2}+O\left(N^{8/9}d^{4/5}\ln^C dN\right),$$

where $\varphi(r)$ is Euler' s function.

With the aid of this theorem one can obtain rather varied information concerning the distribution of the values of the function $L(1,\chi_D)$. For example, from it and from the theorem of Fréchet and Shohat (^{11,12}) there follows a limit theorem for the distribution of the functions (1) in arithmetic progressions whose difference $d\le N^{1/9}$ grows together with N.

As consequences of Theorem 1 one can also obtain information of a different, less usual, nature. Namely, let henceforth d denote the product of the primes not exceeding $q\ge 2$. In this case the following holds.

Theorem 2. For an arbitrary integer $k\ge 1$, uniformly with respect to N, $d\le N^{1/9}$, $1\le l\le d$, $(l,d)=1$, we have

$$\sum_{m=1}^{N}\mu^2(D)L^k(1,\chi_D)=N\prod_{p/d}\left(1-\frac{\chi(l,p)}{p}\right)^{-k}\left[1+O\left(\frac{\ln^2 q}{q}\right)\right].$$

This theorem, in particular, shows that the quantity $\mu^2(D)L(1,\chi_D)$ on the progressions $dm+l$ is, on the average, independent of m and is determined only by the difference of the progression d and its initial term l. It is very plausible that in fact the quantity $L(1, \chi_D)$ on the indicated set of values D is, in the main, always equal to

$$\prod_{p/d}\left(1-\frac{\chi(l,p)}{p}\right)^{-1}.$$

In any case, the following is certainly true.

Theorem 3. Under the conditions of Theorem 2, for each $m = 1, 2, \dots, N$ with $\mu^2(dm + l) = 1$, except for at most $N/\ln^{1-\varepsilon} N$ of them, where $\varepsilon > 0$,

$$L(1, \chi_{dm+l}) = \prod_{p/d}\left(1-\frac{\chi(l,p)}{p}\right)^{-1}\left[1+O\left(\frac{1}{\ln\ln 2d}\right)\right].$$

Hence, with the aid of the known Mertens formulas, one obtains asymptotic expressions for the “large” and “small” values of $L(1, \chi_D)$. Namely, there exist $l = l'$ and $l = l''$ such that, for almost all values D in the sense of Theorem 3, we have

$$L(1, \chi_D) = e^c \ln\ln D + O(1)$$

and, respectively,

$$L(1, \chi_D) = \frac{\pi^2}{6e^c \ln\ln D}\left(1+O\left(\frac{1}{\ln\ln D}\right)\right),$$

where c is Euler’ s constant.

These relations constitute a substantial strengthening of the known inequalities ($^{1-8}$) for $L(1, \chi_D)$, both in the sense of estimating the approximation to the main term of the growth of $L(1, \chi_D)$, and in the sense of the density of the values D for which they hold.

If in the basic lemma one takes $b_D = 1$ when $dm + l$ is a prime number, and $b_D = 0$ otherwise, then from this lemma and from recent results of the “large sieve” in the form of Theorem 8 of (^{13}) there will follow entirely analogous results concerning the values of $L(1, \chi_D)$ on the set of primes $D = dm + l$.

Applications to the theory of quadratic forms and divisors of quadratic fields consist in a simple combination of the estimates given here with the corresponding exact Dirichlet formulas (14) for the number of classes of non-equivalent quadratic forms of discriminant ±D and the number of divisor classes of quadratic fields.

Let, for example, h and R denote, respectively, the class number and the regulator of the quadratic field of discriminant D. Then, for almost all values of D in the sense of Theorem 3, the following asymptotic law holds:

$$\ln hR = \ln \sqrt{|D|} - \ln \prod_{p/d} \left(1 - \frac{\chi(l,p)}{p}\right) - \frac{1+\operatorname{sign} E}{2} \ln 2 - \frac{1-\operatorname{sign} D}{2} \ln \pi + O\left(\frac{1}{\ln\ln 2d}\right).$$

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Note: Figure translations are in progress. See original paper for figures.