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Abstract

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MATHEMATICS

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ON THE CONSTRUCTION OF A PROPER GUIDING FUNCTION FOR A SYSTEM OF DIFFERENTIAL EQUATIONS

(Presented by Academician N. N. Krasovskii, 16 VI 1969)

In recent years M. A. Krasnosel'skii and his students have developed the method of guiding functions for proving the existence of periodic and bounded solutions of systems of ordinary differential equations (see ([1])). Guiding functions have turned out to be a convenient tool for obtaining a priori estimates of periodic, bounded, and almost-periodic solutions of differential equations. From the existence of guiding functions of nonzero index there follows the existence of periodic or bounded solutions for systems of differential equations. Guiding functions have found application in the study of periodic and bounded solutions of equations with delayed argument, etc.

In connection with the above, it is of interest to single out a class of systems of differential equations for which guiding functions exist. For second-order systems with principal homogeneous terms, the most general condition for the existence of a guiding function was obtained by N. A. Bobylev ([2]). In the present paper, existence theorems are proposed for guiding functions for systems of high order with principal homogeneous terms. In constructing guiding functions, a method is used that goes back to N. G. Chetaev and was developed by N. N. Krasovskii ([3]). Applications to theorems on the existence of periodic and bounded solutions of systems of ordinary differential equations and equations with delayed argument are then given.

1. Consider the system of differential equations:

$$dx/dt = F(t,x) \qquad (x \in R^n) \tag{1}$$

with right-hand side continuous in the aggregate of the variables. Following M. A. Krasnosel' skii, we shall call a continuously differentiable function $\varphi(x)$ a **proper guiding function** for system (1) if there exists a continuously differentiable function $\varphi_1(x)$ such that

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$$\lim_{\|x\|\to\infty} \varphi_1(x) = \infty,$$

and a number ρ_0 such that

$$(\operatorname{grad}\varphi(x), F(t,x)) > \|\operatorname{grad}\varphi_1(x)\|\,\|F(t,x)\| \qquad (-\infty < t < \infty,\ \|x\| \geq \rho_0).$$

The vector field $\Phi x = \operatorname{grad}\varphi(x)$ on spheres of radius $\rho \geq \rho_0$ has no zero vectors. Therefore the rotation $\gamma[\Phi;\rho]$ of the field Φ is defined on spheres of radius $\rho \geq \rho_0$, and it does not depend on the choice of ρ. This common rotation $\gamma(\varphi)$ will be called the index of the guiding function φ. Below the following general assertion is used.

Theorem 1 ([1]). *Suppose that for system (1) there exists a proper guiding function of nonzero index.*

Then system (1) has at least one solution $x^*(t)$ uniformly bounded on the entire real axis, and, if the function $F(t,x)$ is ω-periodic in t, then at least one ω-periodic solution $x^*(t)$.

2. The system

$$dx/dt = P(x) \tag{2}$$

will be called an autonomous m-system if $P(x)$ is a positively homogeneous function of order $m > 0$ ($P(\lambda x) = \lambda^m P(x)$ for $\lambda \geq 0$) and on the unit sphere of the space R^n satisfies a Lipschitz condition, while system (2) has no nonzero solutions bounded on the entire real axis $(-\infty, +\infty)$.

Theorem 2. *For there to exist such a continuously differentiable positively homogeneous function $\psi(x)$ that*

$$(\operatorname{grad}\psi(x),\ P(x)) \geq \|x\|^{m+k-1},$$

where k is the order of homogeneity of the function $\psi(x)$, it is necessary and sufficient that (2) be an autonomous m-system.

A statement close to Theorem 2, in the case when the components of the function $P(x)$ are homogeneous polynomials in the components of the vector x, was proved in paper ([4]).

Theorem 2 is used below in constructing a proper guiding function for one class of systems of differential equations.

3. Let the right-hand side $F(t,x)$ of system (1) admit the representation

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$$F(t,x) = P(x) + f(t,x), \tag{3}$$

where $P(x)$ is a positively homogeneous function of order $m > 0$, and $f(t,x)$ is a function continuous in the aggregate of the variables, with

$$\lim_{\|x\|\to\infty} \{\|x\|^{-m} \sup_t \|f(t,x)\|\} = 0. \tag{4}$$

Theorem 3. *Let (2) be an autonomous m-system. Let the vector function $f(t,x)$ satisfy condition (4).*

Then for system (1) there exists a positively homogeneous proper guiding function.

For the case when R^n is two-dimensional ($n = 2$), a statement equivalent to Theorem 3 was proved by N. A. Bobylev ([2]).

Let us note that if the zero solution of system (2) is asymptotically stable in the sense of Lyapunov, then system (2) has no solutions bounded on the entire real axis and, consequently, by virtue of Theorem 3, for system (1), whose right-hand side admits representation (3) and whose function $f(t,x)$ satisfies condition (4), there exists a proper guiding function. However, this fact follows from the well-known theorem of N. N. Krasovskii on the existence of a Lyapunov function.

4. Consider the continuous vector field

$$\Psi x = P(x) \qquad (x \in R^n). \tag{5}$$

If the field Ψ on the unit sphere S has no zero vectors, then its rotation ([5]) $\gamma[\Psi; S]$ on this sphere is defined.

From Theorems 1 and 3 it follows:

Theorem 4. *Let the right-hand side $F(t,x)$ of system (1) be an ω-periodic function in t and admit representation (3), where $f(t,x)$ satisfies condition (4), and $P(x)$ is such that (2) is an autonomous m-system. Finally, let the rotation of the vector field (5) be nonzero: $\gamma[\Psi; S] \neq 0$.*

Then system (1) has at least one ω-periodic solution.

Theorem 5. *Let the right-hand side $F(t,x)$ of system (1) have representation (3), where $f(t,x)$ satisfies condition (4), and the function $P(x)$ is such that (2) is an m-system. Let the rotation $\gamma[\Psi; S]$ of the vector field (5) be nonzero: $\gamma[\Psi; S] \neq 0$.*

Then system (1) has at least one solution bounded on the entire number axis.

We note that in the hypotheses of Theorems 4 and 5 there are neither assumptions on uniqueness of solutions of the system satisfying each initial value, nor assumptions on nonlocal continuability of solutions.

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The computation or estimation of the rotation $\gamma[\Psi, S]$ can in many cases be carried out without difficulty. For example, if $P(-x) = -P(x)$, then $\gamma[\Psi; S]$ is always odd (see, for example, ([1])). Therefore, from Theorems 4 and 5 it follows

Theorem 6. *Let the right-hand side $F(t, x)$ of system (1) have the representation (3), where $f(t, x)$ satisfies condition (4), and let the function $P(x)$ be such that (2) is an autonomous m-system. Finally, let $P(-x) = -P(x)$. Then system (1) has at least one solution bounded on the entire number axis, and if $f(t, x)$ is ω-periodic in t, then at least one ω-periodic solution.*

It follows from Theorem 4 that if the right-hand side of system (1) has the representation (3) and the zero solution of system (2) is asymptotically stable in the sense of Lyapunov, then system (1) has at least one ω-periodic solution (in this case $\gamma[\Psi; S] = (-1)^n$). A special case of this assertion is one result of Cronin ([6]).

4. Let us consider the system of differential equations with retarded argument

$$dx/dt = F[t, x(t), x(t - h_1(t)), \dots, x(t - h_k(t))], \tag{6}$$

where $h_i(t)$ are nonnegative continuous ω-periodic functions, and $F(t, y_0, y_1, \dots, y_k)$ is a function continuous jointly in the variables and ω-periodic in t.

Let the function $F(t, y_0, y_1, \dots, y_k)$ admit the representation

$$F(t, y_0, y_1, \dots, y_k) = P(y_0) + f(t, y_0, y_1, \dots, y_k), \tag{7}$$

where $P(x)$ is a positively homogeneous function of order $m > 0$, and $f(t, y_0, y_1, \dots, y_k)$ satisfies the condition

$$\lim_{\rho \to \infty} \left\{ \rho^{-m} \sup_{t, \|y_j\| \le \rho} \|f(t, y_0, y_1, \dots, y_k)\| \right\} = 0. \tag{8}$$

Theorem 7. *Let the right-hand side $F(t, y_0, y_1, \dots, y_k)$ of system (6) admit the representation (7), where the function $f(t, y_0, y_1, \dots, y_k)$ satisfies condition (8), and let the function $P(x)$ be such that (2) is an autonomous m-system. Suppose that the rotation $\gamma[\Psi; S]$ of the vector field (5) is nonzero: $\gamma[\Psi; S] \neq 0$.*

Then system (6) has at least one ω-periodic solution.

In the proof of this theorem one uses the integro-functional equation (see ([7]))

$$x(t) = x(\omega) + \int_0^t F[(Vx)(s)]\, ds$$

in the space $C[0, \omega]$ of continuous functions, where

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$$(Vx)(t) = x(t + i\omega) \quad (-i\omega \le t < -(i-1)\omega,\ i = 1, 2, ...),$$

$$F[(Vx)(t)] = F(t, (Vx)(t), (Vx)(t - h_1(t)), \dots, (Vx)(t - h_k(t))),$$

whose fixed points coincide with the ω-periodic solutions of system (6).

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Note: Figure translations are in progress. See original paper for figures.

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