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Abstract

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Let $L(A_1 A_2 \dots A_n)$ be a polygonal line with vertices A_k. The turn of the polygonal line L is the number

$$\psi(L) = \sum_k (\pi - \alpha_k),$$

where α_k is the angle between the segments of the polygonal line meeting at the vertex A_k. The variation of turn of a curve is the exact upper bound of the turns of polygonal lines inscribed in this curve. Curves of bounded variation of turn were introduced by A. D. Aleksandrov and have been well studied. Some of their properties that are used below may be found in ([1]). The purpose of the present note is to prove the following theorem.

Theorem. *A homeomorphic point correspondence between convex surfaces that preserves the variations of turn of curves is a similarity.*

Let F and F' be two convex surfaces between which a homeomorphism f is established, preserving the variations of turn of curves. We shall first show that the homeomorphism f is a conformal correspondence. Take on the surface F an arbitrary point A and two directions from this point on the surface. In these directions, draw from the point A curves γ_1 and γ_2 of bounded variation of turn. Such curves are, for example, plane sections of the surface. On the surface F', the point A corresponds to the point A', and to the curves γ_1 and γ_2 correspond the curves γ_1' and γ_2', issuing from the point A'. Since the curves γ_1' and γ_2' are of bounded variation of turn, they have definite directions (semitangents) at the point A'.

Denote by γ the curve composed of the curves γ_1 and γ_2. The variation of its turn is

$$\psi(\gamma) = \psi(\gamma_1) + \psi(\gamma_2) + (\pi - \alpha),$$

where α is the angle between the curves γ_1 and γ_2 at the point A. Similarly, for the curve γ', composed of the curves γ'_1 and γ'_2, we shall have

$$\psi(\gamma') = \psi(\gamma'_1) + \psi(\gamma'_2) + (\pi - \alpha'),$$

where α' is the angle between the curves γ'_1 and γ'_2 at the point A'. By the hypothesis of the theorem, $\psi(\gamma) = \psi(\gamma')$, $\psi(\gamma_1) = \psi(\gamma'_1)$, $\psi(\gamma_2) = \psi(\gamma'_2)$. Therefore $\alpha = \alpha'$. Since the directions of the curves γ_1 and γ_2 at the point A were chosen arbitrarily, this means that the correspondence f is conformal.

Let γ be a closed plane curve on the surface F. It is convex and therefore has variation of turn equal to 2π. Since the corresponding curve on the surface F' is also closed and also has variation of turn 2π, it is plane and convex. Indeed, if it is not plane, then a spatial quadrilateral can be inscribed in it, and its turn is certainly greater than 2π. If it is plane but not convex, then a nonconvex polygon can be inscribed in it, and its turn is greater than 2π.

In order not to burden the proof, let us now restrict ourselves to the case of smooth convex surfaces F and F'. Let A be an arbitrary point of the surface F, and A' the corresponding point of the surface F'. Draw a plane separating the point A from the boundary of the surface F. It divides the surface—

surface F into two parts. Let F_A be that one of these parts to which the point A belongs. Denote by F'_A the corresponding part of the surface F' under the homeomorphism f. Obviously, the surface F'_A also has a plane boundary. In order to prove the similarity of the surfaces F and F', it is enough to prove the similarity of the surfaces F_A and F'_A for an arbitrarily chosen point A. Therefore, without loss of generality, we shall assume that the original surfaces F and F' have plane boundary.

Take two arbitrary points A_1 and A_2 on the surface F. Let A'_1 and A'_2 be the corresponding points on the surface F'. Since the surface F is strictly convex and has a plane boundary, through the points A_1 and A_2 one can draw a plane section γ which does not intersect the boundary of the surface. The corresponding curve γ', passing through the points A'_1 and A'_2 on the surface F', will also be a plane section.

Let σ and σ' be the planes in which the curves γ and γ' lie; let α_1 and α_2 be the tangent planes to the surface F at the points A_1 and A_2; and let α'_1 and α'_2 be the tangent planes to the surface F' at the points A'_1 and A'_2. Construct a projective transformation of space which sends the points A_1 and A_2 to A'_1 and A'_2, the planes α_1 and α_2 to α'_1 and α'_2, and the plane σ to the plane σ'. The last condition is attainable in view of the conformality of the correspondence in the plane pencils with centers A_1 and A'_1, A_2 and A'_2, which is determined by the intersection of the planes σ and σ' with the tangent planes α_1 and α'_1, α_2 and α'_2. This projective transformation has the following properties: 1) it preserves the angles between the lines of the pencil with center A_1 in the plane α_1; 2) it preserves the angles formed by the lines of intersection of the planes

σ with the planes α_1 and α_2. The first property follows from the conformality of the homeomorphism f. The second property follows from the equality of the rotations of the curves γ and γ' between the points A_1 and A_2, A_1' and A_2'.

It is proved that the projective transformation thus constructed is a similarity. Hence it follows that, if the surfaces F and F' are oriented in the same way, then, when they are brought into coincidence at the points A_1 and A_1' and in the corresponding directions at these points, the rays A_1A_2 and $A_1'A_2'$ coincide, while the tangent planes of the surfaces at the points A_2 and A_2' will be parallel.

Bring the surfaces F and F' into coincidence at the points A_1 and A_1' and in the corresponding directions at these points. Take the point of coincidence as the origin of coordinates, and denote by r and r' the vectors of corresponding points of the surfaces F and F' in this position. Then

$$r = \lambda r', \qquad dr = \mu\, dr'.$$

Differentiating the first equality and comparing it with the second, we conclude that $d\lambda = 0$, i.e. $\lambda = \text{const}$, and consequently the surfaces F and F' are similar.

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CITED LITERATURE

[1] A. V. Pogorelov, *Extrinsic Geometry of Convex Surfaces*, "Nauka," 1969.

Note: Figure translations are in progress. See original paper for figures.