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Abstract

Full Text

MATHEMATICS

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EXISTENCE OF WEAK SOLUTIONS OF ONE BOUNDARY-VALUE PROBLEM FOR AN EQUATION OF MIXED TYPE IN MULTIDIMENSIONAL DOMAINS

(Presented by Academician S. L. Sobolev on 2 III 1970)

Let G be a bounded simply connected domain of three-dimensional space $\{x_1, x_2, x_3\}$, bounded by a piecewise smooth surface Γ , and divided by the plane $x_3 = 0$ into two domains $G_1 = G \cap \{x_3 > 0\}$ and $G_2 = G \cap \{x_3 < 0\}$, with $G \cap \{x_3 = 0\}$ a simply connected domain in the plane $\{x_1, x_2\}$. Denote by Σ and S those parts of Γ where $x_3 \geq 0$ and $x_3 < 0$, respectively.

Consider in the domain G the operator

$$Lu = k(x_3)u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} + c(x)u,$$

where $k(x_3)$ is a continuously differentiable function on the interval $[-h_1, h_2]$ ($-h_1 = \inf_{x \in G} x_3$, $h_2 = \sup_{x \in G} x_3$), satisfying the conditions: $k(x_3) > 0$ for $x_3 > 0$, $k(x_3) < 0$ for $x_3 < 0$, $k(0) = 0$, and $k'(x_3) > 0$ on the interval $[-h_1, h_2]$; $c(x)$ is a continuously differentiable function in \bar{G} , with $c(x) \leq 0$ in \bar{G} .

We shall assume that $S = \sum_{i=1}^4 S_i$, where

$$S_1 : x_1 = 1 + \int_0^{x_3} \sqrt{-k(\xi)} d\xi$$

$$S_2 : x_1 = -1 - \int_0^{x_3} \sqrt{-k(\xi)} d\xi, \quad S_3 : x_2 = -1 \quad \text{and} \quad S_4 : x_2 = 1.$$

It is required to find in the domain G a solution of the equation

$$Lu = f, \tag{1}$$

satisfying the boundary condition

$$u = 0 \quad \text{on } \Gamma \setminus S_2. \quad (2)$$

In the works of A. V. Bitsadze^(1,2), some boundary-value problems for equation (1) were first posed and investigated in the case when $k(x_3) = \text{sign } x_3$, $k(x_3) = x_3$, and $c(x) = 0$.

In the present paper* the question of the existence of weak solutions and the uniqueness of a smooth solution of problem (1)–(2) is considered.

Let $C^2(\bar{G}, \Gamma \setminus S_2)$ and $C^2(\bar{G}, \Gamma \setminus S_1)$ be the sets of all twice continuously differentiable functions in the closed domain \bar{G} satisfying, respectively, the conditions $u = 0$ on $\Gamma \setminus S_2$ and $u = 0$ on $\Gamma \setminus S_1$. Denote by $W_2^2(\text{gr})$ and $W_2^2(\text{gr})^+$, respectively, the closures of the sets $C^2(\bar{G}, \Gamma \setminus S_2)$ and $C^2(\bar{G}, \Gamma \setminus S_1)$ in the norm $W_2^2(G)$. Let $f \in L_2(G)$.

By a weak solution of problem (1)–(2) we shall mean a function $u \in L_2(G)$ for which $(u, Lv)_0 = (f, v)_0$ for every $v \in W_2^2(\text{gr})^+$ (by $(\cdot, \cdot)_0$ we denote the scalar product in $L_2(G)$).

As is known⁽³⁾, for the existence of a weak solution of problem (1)–(2) for every $f \in L_2(G)$, it is necessary and sufficient that the inequality

$$\|Lv\|_0 \geq C\|v\|_0, \quad v \in W_2^2(\text{gr})^+, \quad C > 0.$$

* The results of this work were reported at A. V. Bitsadze' s seminar on 16 X 1969.

If the stronger inequality is satisfied,

$$\|Lv\|_0 \geq C\|v\|_1, \quad v \in W_2^2(\Gamma)^+, \quad C > 0, \quad (3)$$

where $\|v\|_1$ is the norm in $W_2^1(G)$, then there exists a weak solution of problem (1)–(2) for every $f \in W_2^{-1}(G)$.

Let $v \in C^2(\bar{G}, \Gamma \setminus S_1)$, and let $p(x), p^i(x)$, $i = 1, 2, 3$, be, for the moment, arbitrary sufficiently smooth functions. Applying Ostrogradskii' s formula, we

obtain

$$\begin{aligned}
 \int_G (pv + p^i v_{x_i}) Lv dx &= \frac{1}{2} \int_G [kp_{x_1 x_1} + p_{x_2 x_2} + p_{x_3 x_3} + 2cp - (cp^i)_{x_i}] v^2 dx + \\
 &+ \frac{1}{2} \int_G \{[-2kp - kp_{x_1}^1 + kp_{x_2}^2 + (kp^3)_{x_3}] v_{x_1}^2 + (-2p + p_{x_1}^1 - p_{x_2}^2 + p_{x_3}^3) v_{x_2}^2 + \\
 &+ (-2p + p_{x_1}^1 + p_{x_2}^2 - p_{x_3}^3) v_{x_3}^2 - 2(p_{x_2}^1 + kp_{x_1}^2) v_{x_1} v_{x_2} - 2(p_{x_3}^1 + kp_{x_1}^3) v_{x_1} v_{x_3} - \\
 &- 2(p_{x_3}^2 + p_{x_2}^3) v_{x_2} v_{x_3}\} dx + \int_\Gamma (kp v v_{x_1} n_1 + p v v_{x_2} n_2 + p v v_{x_3} n_3) ds + \\
 &+ \frac{1}{2} \int_\Gamma (-kp_{x_1} n_1 - p_{x_2} n_2 - p_{x_3} n_3 + cp^i n_i) v^2 ds + \\
 &+ \frac{1}{2} \int_\Gamma [(kp^1 n_1 - kp^2 n_2 - kp^3 n_3) v_{x_1}^2 + (-p^1 n_1 + p^2 n_2 - p^3 n_3) v_{x_2}^2 + \\
 &+ (-p^1 n_1 - p^2 n_2 + p^3 n_3) v_{x_3}^2 + \\
 &+ 2(p^1 n_2 + kp^2 n_1) v_{x_1} v_{x_2} + (p^1 n_3 + kp^3 n_1) v_{x_1} v_{x_3} + 2(p^3 n_2 + p^2 n_3) v_{x_2} v_{x_3}] ds \\
 &= I_1 + I_2 + I_3 + I_4 + I_5,
 \end{aligned} \tag{4}$$

where $n = (n_1, n_2, n_3)$ is the unit vector of the outward normal to Γ (summation over repeated indices from 1 to 3 is assumed).

Consider first I_5 . Since $v = 0$ on $\Gamma \setminus S_1$, it follows that $v_{x_i} = N_v(x) n_i$, $i = 1, 2, 3$, on this surface. On S_1 , $v_{x_1} = n_1 v_n - n_3 v_t$, $v_{x_2} = v_b$, and $v_{x_3} = n_3 v_n + n_1 v_t$, where $t = (\sqrt{-k}/\sqrt{1-k}, 0, -1/\sqrt{1-k})$, $n = (1/\sqrt{1-k}, 0, -\sqrt{-k}/\sqrt{1-k})$, and $b = (0, 1, 0)$. After simple computations we obtain

$$I_5 = \frac{1}{2} \int_{\Gamma \setminus S_1} N_v^2 p^i n_i (kn_1^2 + n_2^2 + n_3^2) ds + \frac{1}{2} \int_{S_1} [(1-k)(-p^1 n_1 + p^3 n_3) v_t^2 + (-p^1 n_1 - p^3 n_3) v_b^2 + 2(1-k) p^2 n_1 n_3 v_t v_b] ds.$$

We choose $p = -\frac{1}{2}$, $p^1 = x_1 - 1 - \varepsilon$, $p^2 = 0$ in \overline{G} , where ε is, for the moment, an arbitrary positive constant; $p^3 = x_3 + \delta$ in \overline{G}_1 , $p^3 = \delta$ in \overline{G}_2 , where $\delta = \text{const} > 0$.

Taking into account that $kn_1^2 + n_2^2 + n_3^2 = 0$ on S_2 and $n_1 = n_3 = 0$ on S_3 and S_4 , we obtain

$$\begin{aligned}
 I_5 &= \frac{1}{2} \int_\Sigma N_v^2 [(x_1 - 1 - \varepsilon) n_1 + (x_3 + \delta) n_3] (kn_1^2 + n_2^2 + n_3^2) ds + \\
 &+ \frac{1}{2} \int_{S_1} \frac{1}{\sqrt{1-k}} [(1-k)(1 - x_1 + \varepsilon - \delta\sqrt{-k}) v_t^2 + \\
 &+ (1 - x_1 + \varepsilon + \delta\sqrt{-k}) v_b^2] ds = I_5' + I_5''.
 \end{aligned}$$

Let $\varepsilon \geq \delta\sqrt{-k}(-h_1)$. Then $I_5'' \geq 0$, since $1 - x_1 \geq 0$ on S_1 . If the surface Σ satisfies the condition

$$\varphi(x) = (x_1 - 1 - \varepsilon) n_1 + (x_3 + \delta) n_3 \geq 0, \tag{5}$$

then $I'_5 \geq 0$. We note that there exist piecewise-smooth surfaces Σ satisfying condition (5). Such, for example, is the surface $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4$, where $\Sigma_1 : x_1 + qx_3 = 1$, $\Sigma_2 : x_1 - qx_3 = -1$, q is a sufficiently large positive constant, $\Sigma_3 : x_2 = -1$, and $\Sigma_4 : x_2 = 1$. Indeed,

$\varphi(x) = 0$ on Σ_3 and Σ_4 , since $n_1 = n_3 = 0$ on these surfaces. On Σ_2 , $n_1 = -1/\sqrt{1+q^2}$, $n_3 = q/\sqrt{1+q^2}$, whence it follows that $\varphi(x) \geq 0$ on Σ_2 . On Σ_1 , $n_1 = 1/\sqrt{1+q^2}$, $n_3 = q/\sqrt{1+q^2}$, and therefore we obtain $\varphi(x) = (-\varepsilon + \delta q)/\sqrt{1+q^2}$. If $q \geq \varepsilon/\delta$, then $\varphi(x) \geq 0$ on Σ_1 .

Obviously, $I_4 \geq 0$. The surface integral I_3 reduces to the double integral

$$I_3 = \frac{1}{4} \int_{D_1} \sqrt{-k} \frac{\partial \bar{v}^2}{\partial x_3} dx_2 dx_3,$$

where D_1 is the projection of S_1 onto the plane of $\{x_2, x_3\}$, and $\bar{v}(x_2, x_3) = v \left[\int_0^{x_3} \sqrt{-k} d\xi, x_2, x_3 \right]$.

Integrating by parts and taking into account that $\bar{v}(x_2, -h_1) = 0$ and $k(0) = 0$, we obtain

$$I_3 = \frac{1}{8} \int_{D_1} \frac{k'}{\sqrt{-k}} dx_2 dx_3 \geq 0.$$

For I_2 we obtain

$$I_2 = \frac{1}{2} \int_{G_1} \{ [k + (x_3 + \delta)k'] v_{x_1}^2 + 3v_{x_2}^2 + v_{x_3}^2 \} dx + \\ + \frac{1}{2} \int_{G_2} (k' \delta v_{x_1}^2 + v_{x_2}^2 + v_{x_3}^2) dx.$$

If the function $c(x)$ satisfies the conditions

$$-3c - (x_1 - 1 - \varepsilon)c_{x_1} - (x_3 + \delta)c_{x_3} \geq 0 \quad \text{for } x \in \bar{G}_1, \quad (6)$$

$$-2c - (x_1 - 1 - \varepsilon)c_{x_1} - \delta c_{x_3} \geq 0 \quad \text{for } x \in \bar{G}_2, \quad (7)$$

then $I_1 \geq 0$. Conditions (6) and (7) are satisfied if, for example, $c = c(x_2)$ in \bar{G} .

Taking into account that $k'(x_3) \geq \inf_{-h_1 < x_3 < h_2} k'(x_3) > 0$, from (4) we obtain

$$\int_G (pv + p_{x_i}^{iv})Lv dx \geq C_1 \int_G \sum_{i=1}^3 v_{x_i}^2 dx, \quad C_1 > 0. \quad (8)$$

Lemma. For functions $v \in \dot{C}^2(\overline{G}, \Gamma \setminus S_1)$, the inequality

$$\int_G v^2 dx \leq C_2 \int_G \sum_{i=1}^3 v_{x_i}^2 dx, \quad C_2 > 0 \quad (9)$$

holds.

From (8) and (9) we obtain

$$C_4 \|Lv\|_0 \|v\|_1 \geq \int_G (pv + p_{x_i}^{iv}) Lv dx \geq C_3 \|v\|_1^2,$$

whence inequality (3) follows for $v \in \dot{C}^2(\overline{G}, \Gamma \setminus S_1)$. By completion we verify the validity of (3) for $v \in \dot{W}_2^2(\text{gr})^+$.

Thus, the following holds.

Theorem 1. If the surface Σ satisfies condition (5), and the function $c(x)$ satisfies conditions (6) and (7), then there exists a weak solution of problem (1)–(2) for any $f \in \dot{W}_2^{-1}(G)$.

It is proved analogously that if Σ satisfies the condition

$$(x_1 + 1 + \varepsilon)n_1 + (x_3 + \delta)n_3 \geq 0, \quad (10)$$

and $c(x)$ satisfies the conditions

$$-3c - (x_1 + 1 + \varepsilon)c_{x_1} - (x_3 + \delta)c_{x_3} \geq 0 \quad \text{for } x \in \overline{G}_1, \quad (11)$$

$$-2c - (x_1 + 1 + \varepsilon)c_{x_1} - \delta c_{x_3} \geq 0 \quad \text{for } x \in \overline{G}_2, \quad (12)$$

then for problem (1)–(2) the inequality

$$\|Lu\|_0 \geq C \|u\|_1, \quad u \in W_2^2(\cdot), \quad C > 0. \quad (13)$$

The functions $p(x)$, $p^1(x)$, and $p^3(x)$ are chosen as above, while $p^1 = x^1 + 1 + \varepsilon$, with $\varepsilon \geq \delta \sqrt{-k(-h_1)}$. We note that the surface Σ constructed above simultaneously satisfies conditions (5) and (10).

By a smooth solution of problem (1)–(2) we shall mean a function $u \in W_2^2(\cdot)$ satisfying equation (1) almost everywhere in the domain G . From inequality (13) it follows that

Theorem 2. If the surface Σ satisfies condition (10), and the function $c(x)$ satisfies conditions (11) and (12), then problem (1)–(2) can have no more than one smooth solution.

Remark 1. Let

$$S = \left(\sum_{i=0}^k S_1^i \right) \cup \left(\sum_{i=1}^{k+1} S_2^i \right) \cup S_3 \cup S_4,$$

where

$$S_1^i : x_1 = a_i + \int_0^{x_3} \sqrt{-k} d\xi,$$

$$S_2^i : x_1 = a_i - \int_0^{x_3} \sqrt{-k} d\xi, \quad 1 = a_0 > a_1 > \dots > a_k > a_{k+1} = -1,$$

$$S_3 : x_2 = -1, \quad S_4 : x_2 = 1.$$

The boundary condition (2) may be replaced by the condition $u = 0$ on

$$\Gamma \setminus \sum_{i=1}^{k+1} S_2^i.$$

Remark 2. Theorems 1 and 2 are also valid in the case when the functions $k(x_3)$ and $c(x)$ have a discontinuity of the first kind on the plane $x_3 = 0$, if $k'(x_3) > 0$ on the intervals $[-h_1, 0]$ and $[0, h_2]$; $k^+ - k^- \geq 0$ and $c^- - c^+ \geq 0$, where $k^\pm = \lim_{x_3 \rightarrow \pm 0} k(x_3)$ and $c^\pm = \lim_{x_3 \rightarrow \pm 0} c(x)$.

Remark 3. Problem (1)–(2) is considered analogously in the case of m independent variables.

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Note: Figure translations are in progress. See original paper for figures.

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