VARIATIONAL THEOREMS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS IN DOUBLY CONNECTED DOMAINS

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Abstract

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VARIATIONAL THEOREMS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS IN DOUBLY CONNECTED DOMAINS

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1. In papers ([1,2]) a general variational theorem was established for classes of analytic functions with a structural formula, and various applications were given of the result obtained to the study of the geometric properties of certain classes of analytic functions defined in the unit disk.

The present paper is devoted to the establishment of theorems on boundary functions for sufficiently general differentiable systems of complex-valued functionals defined on classes of analytic functions in a circular annulus and having a parametric representation by Stieltjes integrals. Thus the solution of concrete extremal problems on such classes of functions is reduced to their investigation only on a certain family of functions depending on a finite number of real parameters.

The standard method of obtaining theorems on boundary functions allows us to restrict ourselves to formulating them only for certain classes of functions, such as the class of functions regular in the annulus $K(q,1) = \{z : q < |z| < 1,\ q > 0\}$ with positive real part, the class of regular univalent and starlike functions in $K(q,1)$, the class of typically real functions in a circular annulus, and the class of functions analytic in the annulus $K(q^2,1)$ and having the integral representation (7).

In proving the theorems formulated and analogous results for other classes of analytic functions defined in doubly connected domains and having a structural formula, we rely essentially on variational theorem 1 of paper ([1]), as well as on the fundamental results of V. A. Zmorovich, L. E. Dunduchenko, Li Yen-Pira, S. A. Kasyanyuk (see ([3,4]) and other authors) devoted to the establishment of integral representations for various classes of analytic functions in a circular annulus.

Examples are given of applications of the results obtained to problems concerning the ranges of values of certain concrete functionals defined on classes of analytic functions in doubly connected domains.

2. Let $w = f(z)$ belong to one of the classes of analytic functions considered below. Fix arbitrarily in the annulus $K(q,1)$ the points $z_1, \dots, z_p$ $(p = 1, 2, \dots)$ and introduce the notation: $u_{mj} = f^{(m)}(z_j)$, $v_{mj} = \bar{u}_{mj}$ $(j = 1, \dots, p;\ m = 0, 1, \dots, s_j)$.

In Theorems 1-4, $I(f) = \{I_n(f)\}$ $(n = 1, \dots, l)$ is a finite system of functionals $I_n(f)$ $(n = 1, \dots, l)$ of the form

$$I_n(f) = J_n(u_{01}, v_{01}, \dots, u_{s_1 1}, v_{s_1 1}; \dots; u_{0p}, v_{0p}, \dots, u_{s_p p}, v_{s_p p}), \tag{1}$$

where J_n $(n = 1, \dots, l)$ are certain analytic functions of their arguments; Q is the order of the system $I(f)$, which is determined by the relation: $Q = s_1 + \dots + s_p$. We note that the system $I(f)$ is assumed to be nonsingular, i.e., the functions $\varphi_k(t)$ from Theorem 1 of ([1]) have a finite number of zeros on the corresponding intervals.

Theorem 1. Let the system $I(f)$ be defined on the class $P(q,1)$ of functions analytic in $K(q,1)$, with positive real part and normalized by the condition

$$\int_0^{2\pi} f(re^{i\varphi})\, d\varphi = 2\pi, \qquad q < r < 1.$$

Then all boundary functions with respect to $I(f)$ are contained in the family

$$f(z) = \sum_{s=1}^{L_1} \mu_1^{(s)} F\left(z e^{-it_1^{(s)}}\right) + \sum_{s=1}^{L_2} \mu_2^{(s)} F\left(\frac{q}{z} e^{it_2^{(s)}}\right) - 1, \tag{2}$$

where

$$F(x) = \frac{1+x}{1-x} + 2\sum_{k=1}^{\infty} \frac{q^{2k}}{1-q^{2k}} \left(x^k - x^{-k}\right). \tag{3}$$

Here $\mu_k^{(s)}$, $t_k^{(s)}$ $(k = 1, 2)$ are real parameters subject to the conditions: $\mu_k^{(s)} \geq 0$, $\mu_k^{(1)} + \dots + \mu_k^{(L_k)} = 1$, $-\pi < t_k^{(1)} < \dots < t_k^{(L_k)} \leq \pi$. Moreover $L_k \leq Q + 2p$ $(k = 1, 2)$.

Denote by $S^*(q,1)$ the class of regular univalent and starlike functions $w = f(z)$ in the annulus $K(q,1)$ ([5]).

Theorem 2. Let the system $I(f)$ be defined on the class $S^*(q,1)$. Then all boundary functions with respect to $I(f)$ are contained in the family

$$f(z) = cz \prod_{s=1}^{L_2} \left\{ \prod_{k=1}^{\infty} \left(1 - q^{2k-1} z e^{-it_2^{(s)}}\right) \left(1 - q^{2k-1} z^{-1} e^{it_2^{(s)}}\right) \right\}^{\mu_2^{(s)}} \times$$
$$\times \prod_{s=1}^{L_1} \left\{ (1 - z e^{-it_1^{(s)}}) \prod_{k=1}^{\infty} \left(1 - q^{2k} z e^{-it_1^{(s)}}\right) \left(1 - q^{2k} z^{-1} e^{it_1^{(s)}}\right) \right\}^{\mu_1^{(s)}-1}, \tag{4}$$

$$c = \text{const},$$

depending on a finite number of real parameters $\mu_k^{(s)}$, $t_k^{(s)}$, subject to the conditions: $\mu_k^{(s)} \geq 0$, $\mu_k^{(1)} + \dots + \mu_k^{(L_k)} = 2$, $-\pi < t_k^{(1)} < \dots < t_k^{(L_k)} \leq \pi$. Moreover $L_k \leq Q + 2p$ $(k = 1, 2)$.

Functions of the form (4) map the annulus $K(q, 1)$ onto starlike doubly connected domains obtained from the whole plane by deleting L_1 rays going to infinity and containing the origin on their prolongations, and L_2 segments issuing from the origin.

Denote by $T_r(q, 1)$ the class of typically real functions

$$f(z) = \dots + \frac{c_{-1}}{z} + c_0 + c_1 z + \dots,$$

regular in the annulus $K(q, 1)$, and by $T_q(c_{-1}, c_1)$ the subclass of this class with fixed coefficients c_k $(k = 0, \pm 1)$ ([6]).

Theorem 3. Let the system of functionals $I(f)$ be defined on the class $T_q(c_{-1}, c_1)$. Then all boundary functions with respect to $I(f)$ are contained in the family

$$f(z) = (c_1 - c_{-1}) \sum_{s=1}^{L_1} \mu_1^{(s)} S_q\left(z, \tau_1^{(s)}\right) - (qc_1 - q^{-1} c_{-1}) \sum_{s=1}^{L_2} \mu_2^{(s)} S_q\left(\frac{q}{z}, \tau_2^{(s)}\right) + c_0, \tag{5}$$

where

$$S_q(x, \tau) = \sum_{k=-\infty}^{\infty} \frac{x q^{2k}}{1 - 2x q^{2k} \tau + x^2 q^{4k}}. \tag{6}$$

Here the real parameters $\mu_k^{(s)}, \tau_k^{(s)}$ are subject to the conditions: $\mu_k^{(s)} \geq 0$, $\mu_1^{(1)} + \dots + \mu_1^{(L_1)} = 1$, $\mu_2^{(1)} + \dots + \mu_2^{(L_2)} = 1$, $-1 \leq \tau_k^{(1)} < \dots < \tau_k^{(L_k)} \leq 1$, and $L_k \leq 2(Q + 2p)$ $(k = 1, 2)$.

Denote by U_q^* the class of functions analytic in the annulus $K(q^2, 1)$ which have the integral representation (4)

$$f(z) = z \exp\left\{-2\alpha \int_{-\pi}^{\pi} \ln\left[\frac{1 - ze^{-it}}{1 - q^2 z^{-1} e^{it}}\right] d\mu(t)\right\}, \tag{7}$$

where $0 < \alpha \leq 1$, and $\mu(t)$ is a nondecreasing function on $[-\pi, \pi]$, normalized by the conditions $\mu(-\pi + 0) = \mu(-\pi) = 0$, $\mu(\pi) = 1$. The class U_q^* of functions $w = f(z)$ symmetric with respect to the circle $|w| = q$ is a natural generalization of the well-known class of functions starlike in the unit disk of order α.

Theorem 4. Let a system of functionals $I(f)$ be defined on the class U_q^*. Then all boundary functions with respect to $I(f)$ are contained in the family

$$f(z) = z \cdot \prod_{k=1}^{N} \left(\frac{1 - q^2 z^{-1} e^{it_k}}{1 - ze^{-it_k}}\right)^{\alpha\mu_k}, \tag{8}$$

depending on a finite number of real parameters μ_k, t_k $(k = 1, \dots, N)$, connected by the conditions: $\mu_k \geq 0$, $\mu_1 + \dots + \mu_N = 2$, $-\pi < t_1 < \dots < t_N \leq \pi$; moreover $N \leq 2(Q + p)$.

Similar theorems on boundary functions hold for finite systems composed of functionals analytically depending on the coefficients of the expansion of the function $w = f(z)$ in a Laurent series in the annulus $K(q, 1)$, and also for systems of mixed type. We give one of them, proved earlier by another method by Yu. E. Alenitsyn [7].

Theorem 5. Let the system of functionals $I(f) = \{c_{m_1}, \dots, c_{m_N}\}$ $(m_1 < m_2 < \dots < m_N)$ be defined on the class $T_q(c_{-1}, c_1)$. Then, if $|m_k| \neq |m_l|$ for $k \neq l$, all boundary functions with respect to $I(f)$ have the form (5), with $L_k \leq [\frac{n+1}{2}]$, where $n = \max(|m_1|, |m_N|)$.

3. We give several examples of applications of the theorems of the preceding paragraph to the solution of concrete extremal problems.

Theorem 6. The domain D of values of the system of functionals

$$I(f) = \{f(z), f'(z), \dots, f^{(n)}(z)\} = \{w_1, \dots, w_{n+1}\} \tag{9}$$

(z belongs to the annulus $K(q, 1)$ and is fixed), defined on the class $P(q, 1)$, is a closed bounded set coinciding with the convex hull of the geometric sum of two sets in $(n + 1)$-dimensional complex space

$$W_{l+1} = F^{(l)}(ze^{it}), \qquad l = 0, 1, \dots, n; \quad -\pi < t \leq \pi,$$

$$W_{l+1} = F^{(l)}\left(\frac{q}{z} e^{-it}\right) - a_l, \qquad l = 0, 1, \dots, n; \quad -\pi < t \leq \pi, \tag{10}$$

where $a_l = 1$ for $l = 0$; $a_l = 0$ for $l \neq 0$.

The boundary functions are functions of the form (2), with $L_k \leq n + 2$.

Theorem 7. The domain D of values of the functional

$$I(f) = \ln \frac{f(z)}{cz}, \qquad |z| = r, \tag{11}$$

defined on the class $S^*(q, 1)$, is the convex hull of the geometric sum of two plane arcs

$$I = -2\left\{\ln(1 - re^{it}) + \sum_{k=1}^{\infty} \ln(1 - q^k re^{it})(1 - q^{2k} r^{-1} e^{-it})\right\},$$

$$I = 2\sum_{k=1}^{\infty} \ln(1 - q^{2k-1} re^{it})(1 - q^{2k-1} r^{-1} e^{-it}), \tag{12}$$

$$-\pi < t \leq \pi.$$

The boundary functions are functions of the form (4), with $L_k \leq 2$. From this theorem, in particular, follow the sharp estimates for the modulus and argument of the function $f(z) \in S^*(q, 1)$ obtained earlier by L. E. Dunduchenko ([8]).

Theorem 8. The domain D of values of the functional

$$I(f) = \ln \left|\frac{f(z)}{z}\right| + i \operatorname{Re} \frac{zf'(z)}{f(z)} = x + iy. \tag{13}$$

defined on the class U_q^*, is a closed set coinciding with the convex hull of the curve

$$y = \frac{a}{a\tilde{b} - ab}(ce^{x/a} - \tilde{b})(1 + bc^{-1}e^{-x/a}) + 1 - 2a, \tag{14}$$

where

$$a - b \leq x \leq a + b, \qquad a(r) = \frac{1 + r^2}{1 - r^2}, \qquad b(r) = \frac{2r}{1 - r^2},$$

$$\tilde{a} = a(q^2/r), \qquad \tilde{b} = b(q^2/r), \qquad c = \frac{2r^3}{r^2 - q^4}.$$

The boundary functions are functions of the form (8), where $N \leq 2$.

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Note: Figure translations are in progress. See original paper for figures.