ON SOME EMBEDDING THEOREMS FOR \(L_p\)-CLASSES OF FUNCTIONS

MATHEMATICS

1970

SovietRxiv

Abstract

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UDC 517.512.6

MATHEMATICS

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ON SOME EMBEDDING THEOREMS FOR L_p-CLASSES OF FUNCTIONS

(Presented by Academician S. L. Sobolev on 14 I 1970)

Let L_p be the class of measurable periodic functions $f(x)$ of period 2π satisfying the condition

$$\int_0^{2\pi} |f(x)|^p \, dx < \infty.$$

For each function $f(x)$ with Fourier series

$$\sum_{n=1}^{\infty} A_n(x) = \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \tag{1}$$

consider the quantity

$$E_n(f)_{L_p} = \inf_{\alpha_k, \beta_k} \left\{ \int_0^{2\pi} \left| f(x) - \sum_{k=0}^{n} \alpha_k \cos kx + \beta_k \sin kx \right|^p dx \right\}^{1/p}$$

and, for every integer $k \geq 0$, the quantity

$$\omega_k(f; h)_{L_p} = \sup_{|t| \leq h} \left\| \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu} f(x + \nu t) \right\|_{L_p} \qquad (k \geq 1),$$

$$\omega_0(f; h)_{L_p} = \|f(x)\|_{L_p} = \left\{ \int_0^{2\pi} |f(x)|^p \, dx \right\}^{1/p}.$$

The following theorem of Hardy–Littlewood–Paley is known. Suppose that for $2 < p < \infty$ the series

$$N_p(f) = \sum_{n=1}^{\infty} (|a_n|^p + |b_n|^p) \, n^{p-2}$$

converges. Then

$$\|f(x)\|_{L_p}^p \leq M_p N_p(f). \tag{2}$$

Since always

$$|a_n| + |b_n| \leq E_{n-1}(f)_{L_1} \leq C_k \omega_k(f; 1/n)_{L_1},$$

it follows immediately from inequality (2) that, for $2 < p < \infty$,

$$\|f(x)\|_{L_p} \leq C_p \left\{ \sum_{n=1}^{\infty} E_{n-1}^p(f)_{L_1}\, n^{p-2} \right\}^{1/p} \leq C_{p,k} \left\{ \sum_{n=1}^{\infty} \omega_k^p \left(f; \frac{1}{n}\right)_{L_1} n^{p-2} \right\}^{1/p}. \tag{3}$$

From the same Hardy–Littlewood–Paley theorem, formulated in the form of the inequality

$$\|f(x)\|_{L_p}^p \leq C_p^p \sum_{n=1}^{\infty} (a_n^* + b_n^*)^p n^{p-2} \qquad (2 < p < \infty), \tag{4}$$

where $\{a_n^* + b_n^*\}$ denotes the sequence of numbers obtained from the sequence

$$\rho_n = \sqrt{|a_n|^2 + |b_n|^2}$$

after arranging it in decreasing order, it follows, for any r $(1 < r \leq 2)$, that an inequality of the form

$$\|f(x)\|_{L_p}^p \leq C_{p,r}^p \sum_{n=1}^{\infty} E_{n-1}^p(f)_{L_r}\, n^{p/r-2}. \tag{5}$$

Indeed, since $1 < r \leqslant 2$, by the theorem of Hardy–Littlewood–Paley (see (^{2}), p. 185) we have

$$\left\{ \sum_{n=1}^{\infty} (a_n^* + b_n^*)^r n^{r-2} \right\}^{1/r} \leqslant C_r \|f(x)\|_{L_r}.$$

Applying this inequality to the difference $f(x) - S_{n-1}(f; x)$, where $S_{n-1}(f; x)$ is the partial sum of order $n-1$ of the series (1), we obtain

$$\left\{ \sum_{\nu=n}^{2n} (a_\nu^* + b_\nu^*)^r \nu^{r-2} \right\}^{1/r} \leqslant C_r E_{n-1}(f)_{L_r}.$$

By the monotonicity of $\{a_n^* + b_n^*\}$, it follows from this that

$$a_{2n}^* + b_{2n}^* \leqslant C_r E_{n-1}(f)_{L_r}\, n^{1/r-1}. \tag{6}$$

This inequality, together with (4), gives (5) for $p > 2$.

In the case when $1 < p \leqslant 2$, the embedding theorem from L_r into L_p of the type of inequalities (3) and (5), as was noted earlier by the author (see (^{10}), Theorems 1.3.9, 1.4.11, $k=0$), can be obtained with the aid of the well-known Littlewood–Paley theorem (see (^{2}), p. 348).

By this theorem (see (^{9}), p. 128), for every p $(1<p\leqslant 2)$

$$\|f(x)\|_{L_p}\leqslant M_p\left\{\sum_{m=0}^{\infty}\left\|\sum_{\nu=2^m}^{2^{m+1}-1}A_\nu(x)\right\|_{L_p}^{p}\right\}^{1/p}. \tag{7}$$

Applying here the inequality of S. M. Nikol' skii (see (^{1}), p. 256), we have

$$\|f(x)\|_{L_p}\leqslant 2M_p\left\{\sum_{m=0}^{\infty}\left\|\sum_{\nu=2^m}^{2^{m+1}-1}A_\nu(x)\right\|_{L_r}^{p}2^{(m+1)(1/r-1/p)p}\right\}^{1/p}\leqslant$$

$$\leqslant M_{p,r}\left\{\sum_{m=0}^{\infty}E^p_{2^m-1}(f)_{L_r}\,2^{m(p/r-1)}\right\}^{1/p}\leqslant M'_{p,r}\left\{\sum_{n=1}^{\infty}E^p_{n-1}(f)_{L_r}\,n^{p/r-2}\right\}^{1/p}. \tag{8}$$

If $r=1$, then, choosing an arbitrary γ such that $1<\gamma<p$, from (8) we have the estimate

$$\|f(x)\|_{L_p}\leqslant M_p\left\{\sum_{n=1}^{\infty}E^p_{n-1}(f)_{L_\gamma}\,n^{p/\gamma-2}\right\}^{1/p},$$

after applying to it the inequality of Konyushkov–Stechkin (see (^{3}), p. 56)

$$E_n(f)_{L_\gamma}\leqslant M\left\{E_n(f)_{L_1}\,n^{1-1/\gamma}+\sum_{\nu=n}^{\infty}\nu^{-1/\gamma}E_\nu(f)_{L_1}\right\}$$

and the known numerical inequality (see (^{4}), p. 308)

$$\sum_{n=1}^{\infty}n^{-c}(d_n+d_{n+1}+\ldots)^p\leqslant M\sum_{n=1}^{\infty}n^{-c}(nd_n)^p\qquad(p<1,\ c<1,\ d_n\geqslant 0)$$

we obtain

$$\|f(x)\|^p_{L_p}\leqslant M'^{\,p}_p\left\{\sum_{n=1}^{\infty}E^p_{n-1}(f)_{L_1}n^{p-2}+\sum_{n=1}^{\infty}n^{p/\gamma-2}\left(\sum_{\nu=n}^{\infty}E_\nu(f)_{L_1}\nu^{-1/\gamma}\right)^p\right\}\leqslant$$

$$\leqslant M_p''^p \sum_{n=1}^{\infty} E_{n-1}^p(f)_{L_1} n^{p-2}.$$

Examples of functions

$$f_0(x) \sim \sum_{m=0}^{\infty} \delta_m(r) \sum_{\nu=2^m}^{2^{m+1}-1} \cos \nu x, \qquad g_0(x) \sim \sum_{m=0}^{\infty} \delta_m(r) \sum_{\nu=2^m}^{2^{m+1}-1} \sin \nu x,$$

$$\delta_m(r) = \left[2^{-m(r-1)}\left(a_{2^m}^r - a_{2^{m+1}}^r\right)\right]^{1/r}, \qquad a_n \downarrow 0, \qquad 1 < r < \infty,$$

for which

$$E_n(f_0)_{L_r} \leq M_r a_n, \qquad E_n(g_0)_{L_r} \leq M_r a_n,$$

show that inequalities (5) and (8) cannot be improved.

Let us note that from Jackson’ s theorem and the author’ s inequality (see $(^9)$, Theorem 1)

$$\omega_k\left(f; \frac{1}{n}\right)_{L_p} \leq \frac{c_{p,k}}{n^k}\left\{\sum_{\nu=1}^{n} \nu^{kl-1} E_{\nu-1}^l(f)_{L_p}\right\}^{1/l} \qquad (l = \min(2, p)) \qquad (9)$$

it follows that estimates (3), (5), and (8) are equivalent to the inequality

$$\|f(x)\|_{L_p} \leq C_p \left\{\sum_{n=1}^{\infty} \omega_k^p\left(f; \frac{1}{n}\right)_{L_r} n^{p/r-2}\right\}^{1/p}. \qquad (10)$$

In the general case $1 \leq r < p < \infty$, inequalities of type (10) for $k = 1$, as was shown by P. L. Ul’ yanov (see $(^5)$), can be obtained by relying on considerations connected with the notion of an equimeasurable function. P. L. Ul’yanov showed, moreover, that inequality (10) ($k = 1$) cannot, in a certain sense, be improved. Let us note that instead of the zero Fourier coefficient, which here is taken to be zero, in P. L. Ul’ yanov’ s work the right-hand side of inequality (10) ($k = 1$) contains the additional term $\|f(k)\|_{L_1}$.

We have dwelt in detail on the considerations given above not only in order to show how embedding theorems of the type of inequalities (3), (5), and (8), in the corresponding cases, follow from the Hardy-Littlewood-Pólya theorems, which was already noted by the author in $(^{10})$ (see $(^{10})$, Theorems 1.3.9, 1.4.11, $k = 0$)*; but chiefly in order to show the method for obtaining, in these cases, embedding theorems of this type for L_p-classes of functions of many variables**.

For any p $(1 < p < \infty)$ and $1 \le r \le 2$ $(r < p)$, the following assertions are valid, formulated in the notation adopted in the author’ s paper (^{11}).

Theorem 1. If* $f(x_1, \dots, x_k) \in L_r^{(k)}$ *and the conditions*

$$E^{(\nu)}(f;p;r) = \left\{ \sum_{n=1}^{\infty} \left(E_{n-1,\infty}^{(\nu)}(f)_{L_r^{(k)}} \right)^p n^{(p/r-1)k-1} \right\}^{1/p} < \infty \qquad (\nu = 1, 2, \dots, k),$$

or the equivalent conditions

$$\Omega_m^{(\nu)}(f;p;r) = \left\{ \sum_{n=1}^{\infty} \left(\omega_m^{(\nu)} \left(f; \frac{1}{n} \right)_{L_r^{(k)}} \right)^p n^{(p/r-1)k-1} \right\}^{1/p} < \infty \qquad (\nu = 1, 2, \dots, k),$$

$$\left(m > k \left(\frac{1}{r} - \frac{1}{p} \right) \right), \quad \text{then } f(x_1, \dots, x_k) \in L_p^{(k)},$$

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* For arbitrary functions defined on the whole real axis, see also (^{8}). In (^{5}) and in the later paper (^{7}) there are no references to these works, nor to the preceding work of the author (^{9}), in which, in the more general case, the exact order inequalities between moduli of smoothness and best approximations for the classes L_p, needed in passing from inequalities of type (10) to inequalities (3), (5), and (8), were obtained by the method indicated above.

** Attention is drawn in (^{5}) to the well-known interest of finding assertions of this type for functions of many variables (see (^{5}), p. 685).

*** In Theorems 1 and 2 the mean value of the function $f(x_1, \dots, x_k)$ over the period is assumed to be zero.

$$\|f\|_{L_p^{(k)}} = \|f(x_1, \dots, x_k)\|_{L_p^{(k)}} \le C_{p,k} \sum_{\nu=1}^{k} E^{(\nu)}(f;p;r)$$

$$\|f\|_{L_p^{(k)}} \le C_{p,k} \sum_{\nu=1}^{k} \Omega^{(\nu)}(f;p;r). \tag{11}$$

An example of a function $f(x_1, \dots, x_k) \in L_r^{(k)}$ with Fourier series

$$\sum_{l=0}^{\infty} \delta_l(k;r) \prod_{i=1}^{k} \sum_{\nu_i=2^l}^{2^{l+1}-1} \begin{matrix} \cos \nu_i x \\ \sin \nu_i x \end{matrix}, \qquad \text{where } \delta_l(k;r) = 2^{-lk(1-1/r)} (\alpha_{2^l}^r - \alpha_{2^{l+1}}^r)^{1/r}, \quad \alpha_n \downarrow 0,$$

shows that inequality (11) cannot be improved.

We note that Theorem 1 contains k conditions ensuring the embedding from $L_r^{(k)}$ into $L_p^{(k)}$. Here the rate of convergence to zero of the quantities $E_{n,\infty}^{(\nu)}(f)_{L_r^{(k)}}$ for all ν $(\nu = 1, 2, \dots, k)$ must be of order $o(n^{1/p-1/r})$ and such that the series $E^{(\nu)}(f; p; r)$ converge. In this connection we give one more assertion, of a different character, in which the "weak" properties of functions with respect to some variables, in order to ensure the embedding from $L_r^{(k)}$ into $L_p^{(k)}$, may be compensated by stronger properties with respect to other variables.

Theorem 2. Let $f(x_1, \dots, x_k) \in L_r^{(k)}$, and suppose that for some system of numbers $\alpha_\nu > 0$ $(\nu = 1, 2, \dots, k;\ \alpha_1 + \dots + \alpha_k = 1)$ the conditions

$$\widetilde{E}^{(\nu)}(f; p; r) = \left\{ \sum_{n=1}^{\infty} \left(E_{n-1,\infty}^{(\nu)}(f)_{L_r^{(k)}} \right)^{\alpha_\nu p} n^{p/r-2} \right\}^{1/p} < \infty \qquad (\nu = 1, 2, \dots, k)$$

or the equivalent conditions

$$\widetilde{\Omega}_{m_\nu}^{(\nu)}(f; p; r) = \left\{ \sum_{n=1}^{\infty} \left(\omega_{m_\nu}^{(\nu)} \left(f; \frac{1}{n} \right)_{L_r^{(k)}} \right)^{\alpha_\nu p} n^{p/r-2} \right\}^{1/p} < \infty$$

$$\left(\nu = 1, 2, \dots, k;\ m_\nu > \left(\frac{1}{r} - \frac{1}{p} \right) \frac{1}{\alpha_\nu} \right).$$

Then $f(x_1, \dots, x_k) \in L_p^{(k)}$,

$$\|f\|_{L_p^{(k)}} \le C_{p,k} \prod_{\nu=1}^{k} \widetilde{E}^{(\nu)}(f; p; r), \qquad \|f\|_{L_p^{(k)}} \le C_{p,k} \prod_{\nu=1}^{k} \widetilde{\Omega}_{m_\nu}^{(\nu)}(f; p; r).$$

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Received
29 XII 1970

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Note: Figure translations are in progress. See original paper for figures.

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