ON SOME PROPERTIES OF \(R\)-, \(R^c\)-OPERATIONS AND PROJECTIVE OPERATIONS ON UNCOUNTABLE SYSTEMS OF SETS IN CONNECTION WITH BRANCHING HYPOTHESES

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I. D. STUPINA

ON SOME PROPERTIES OF R-, R^c-OPERATIONS AND PROJECTIVE OPERATIONS ON UNCOUNTABLE SYSTEMS OF SETS IN CONNECTION WITH BRANCHING HYPOTHESES

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In the paper (1) we considered certain properties of R- and R^c-operations on countable systems of sets. Analogous properties of certain operations were considered in (2). In the present note analogous questions are considered for the same operations on uncountable systems of sets. We use the notation introduced in note (1).

1. Let $T = \langle \mathcal{E}, < \rangle$ be a branching table, $U \subset \mathcal{E}$, $T^* = \langle U(T), < \rangle$. If $\rho(T^*) = \omega_\lambda$, ω_τ is the final character of the number ω_λ, and $F = (x_\beta)_{\beta < \omega_\lambda}$ is a monotone dissection of the table T', then $(\forall \beta < \omega_\lambda) \cdot [\overline{(x_\beta, \cdot)_{T^*} \cap U} \geq \aleph_\tau]$, and further either $\overline{F} \cap \overline{U} = \aleph_\tau$, or there exists a disjunctive subset $K \subset U$ such that $(\forall \beta < \omega_\lambda)[\overline{(x_\beta, \cdot)_{T^*} \cap K} \geq \aleph_\tau]$.

In what follows, by $\aleph_\nu$ we shall denote a strongly inaccessible cardinal number. If $\overline{E} = p$, then E will be called a p-set. The following hypotheses are used in the paper:

(α_1) Every branching table T of rank ω_ν such that $(\forall \alpha < \omega_\nu) \cdot [\overline{T_\alpha} < \aleph_\nu]$ attains its rank.

(α_2) If in a branching table $T = \langle \mathcal{E}, < \rangle$ of rank ω_ν the cardinality of every disjunctive subset $U \subset \mathcal{E}$ is less than $\aleph_\nu$, then the table T attains its rank (3).

These hypotheses are respectively equivalent to the following:

(β_1) Every branching table T of rank ω_ν either attains its rank or has an $\aleph_\nu$-node.

(β_2) Every branching table $T = \langle \mathcal{E}, < \rangle$ of rank ω_ν either attains its rank or satisfies the condition: there exists a disjunctive $\aleph_\nu$-subset $U \subset \mathcal{E}$ not satisfying condition i_ν.

Obviously, $(\alpha_1) \Rightarrow (\alpha_2)$. Theorems proved with the aid of the hypotheses (α_1) and (α_2) will be marked by $(*)$ and $(**)$, respectively.

$(*)$ For every $\aleph_\nu$-set $U \subset \mathcal{E}$ in the table $T = \langle \mathcal{E}, < \rangle$ of rank $\leq \omega_\nu$, I_ν or II_ν is valid.

2. By the Π^τ-product of discrete spaces $\{X_\alpha : \alpha < \omega_\tau\}$, denoted by us as $\Pi^\tau X_\alpha$ and called a generalized Baire space, we shall mean the space on the abstract product

$$\prod_{\alpha<\omega_\tau} X_\alpha,$$

whose open-closed base B^τ is given by the totality of generalized Baire intervals $\delta_{(i_\beta)_{\beta<\alpha}}$, where

$$\delta_{(i_\beta)_{\beta<\alpha}} = \left\{ (j_\beta)_{\beta<\omega_\tau} \in \prod_{\beta<\omega_\tau} X_\beta : (\forall \beta < \alpha)\, [j_\beta = i_\beta] \right\}.$$

We shall assume

$$\delta_{(i_\beta)_{\beta<0}} = \prod_{\alpha<\omega_\tau} X_\alpha.$$

If $(\forall \alpha < \omega_\tau)\, [X_\alpha = J]$ and $J \neq \aleph_\tau$, then the space $\Pi^\tau X_\alpha$ is denoted by J^{ω_τ}. Put $T^\tau = \langle B^\tau, < \rangle$, where as the relation $<$ the relation $\supset$ of strict inclusion is taken.

$(*)$ In order that a closed set $E \subseteq \Pi^\tau X_\alpha$ not be $\aleph_\nu$-bicompact, it is necessary and sufficient that it have a disjunctive

an $\aleph_\nu$-covering $S \subset B^\nu$, satisfying condition II_ν, i.e., such that the branching table $\langle S(T^\nu), < \rangle$ has an $\aleph_\nu$-node.

Remark. I. I. Parovichenko ([4]) proved that hypothesis (a_1) is equivalent to the assertion of $\aleph_\nu$-bicompactness of the T^ν-product of discrete spaces $\{X_\alpha : \alpha < \omega_\nu\}$, where $(\forall \alpha < \omega_\nu)[\overline{\overline{X}}_\alpha < \aleph_\nu]$. Since the classes of open sets in this space and in the space $\prod^\nu X_\alpha$ coincide, hypothesis (a_1) is equivalent to the assertion of $\aleph_\nu$-bicompactness of the space $\prod^\nu X_\alpha$, when $(\forall \alpha < \omega_\nu)[\overline{\overline{X}}_\alpha < \aleph_\nu]$.

3. The set of all tuples of the form $(i_\beta)_{\beta<\alpha}$, corresponding to all points $x \in J^\nu$, will be denoted by W. A set $U \subset W$ will be called a W-base. Unless special qualifications are made, the R- and R^c-operations are considered with the full depth of chains $\upsilon = \omega_\nu$. The rigid W-bases of the operations $R_{\mathfrak{M}}$, $R^c_{\mathfrak{M}}$, R^α_N, $R^{\alpha c}_N$, R^α, $R^{\alpha c}$, $0 \leqslant \alpha < \omega_{\nu+1}$ ([5], [6]) will be denoted respectively by $\theta_{\mathfrak{M}}$, $\theta^c_{\mathfrak{M}}$, θ^α_N, $\theta^{\alpha c}_N$, θ^α, $\theta^{\alpha c}$. By χ_α $[\chi^c_\alpha]$ we denote the rigid base of the projective A_α $[CA_\alpha]$-operation, $0 \leqslant \alpha < \omega_{\nu+1}$.

4. We shall say that a rigid base N admits a V-transformation if, for every $J' \subset J$, under the condition $N^{J'} \neq \emptyset$, there exists a set $[J']$ such that, putting for an arbitrary system of sets (E_i) $E_i = E'_i$, if $i \notin [J']$, and $E_i = \emptyset$, if $i \in [J']$, we obtain $\Phi_N^{J'}(E_i) = \Phi_N(E_i)$.

We have proved: if N and N^c are rigid bases of mutually complementary operations, then each of them admits a V-transformation.* Therefore the bases χ_α, χ_α^c, $0 \leqslant \alpha < \omega_{\nu+1}$, and also the bases θ_N^α, $\theta_N^{\alpha c}$, $0 \leqslant \alpha < \omega_{\nu+1}$, under the condition that N and N^c are rigid bases, admit a V-transformation.

A rigid base N is called $\aleph_\tau$-regular (8) for a class of sets $\mathscr{K}$ if $(\forall J' \subset J)[\Phi_N^{J'}(\mathscr{K}) \subset \Phi_N(\mathscr{K})]$ and the class of sets $\Phi_N(\mathscr{K})$ is invariant with respect to the operations $\bigcup\limits_{\aleph_\tau}$, $\bigcap\limits_{\aleph_\tau}$. A property H of chains of a rigid base N is called N-regular (8) for a class of sets $\mathscr{K}$ if the operations Φ_{HN}, $\Phi_{(HN)^i}$ are weaker than the operation Φ_N with respect to the class of sets $\mathscr{K}$.

Each of the bases χ_α, χ_α^c, $0 \leqslant \alpha < \omega_{\nu+1}$, and also each of the bases θ_N^α, $\theta_N^{\alpha c}$, $1 \leqslant \alpha < \omega_{\nu+1}$, when the conditions are fulfilled: 1°. N and N^c are rigid bases; 2°. The operation Φ_N is stronger than the operation $\bigcup\limits_{\aleph_\nu}$, is $\aleph_\nu$-regular for the class of sets $\mathscr{K} \ni \emptyset, \Xi$, where Ξ is the basic space.

Hence, and by virtue of Theorem 3 of I. Kozlova ((8), Theorem 1), for the class of sets $\mathscr{K} \ni \emptyset, \Xi$, for every $p \leqslant \aleph_0$ the property H_x is M-regular, if M is a base χ_α, χ_α^c, $0 \leqslant \alpha < \omega_{\nu+1}$, and also a base θ_N^α, $\theta_N^{\alpha c}$, $1 \leqslant \alpha < \omega_{\nu+1}$ in the case when conditions 1° and 2° are fulfilled, and, consequently, when M is a base θ^α, $\theta^{\alpha c}$, $1 \leqslant \alpha < \omega_{\nu+1}$.

5. For a base N define the $\Delta\Sigma$-operation Q_N over two systems of sets (E_u), (e_v), by putting

$$Q_N(E_u, e_v) = \bigcup_{u,v} \bigcup_{\xi \in N,\ \xi' \subset \xi,\ \overline{\overline{\xi'}} = \aleph_\nu} \left(\bigcap_{u \in \xi} E_u \cap \bigcap_{v \in \xi'} e_v \right).$$

(*) For any $0 \leqslant \alpha < \omega_{\nu+1}$ the operation Q_{θ^α} $[Q_{\theta^{\alpha c}}]$ is weaker than the operation Φ_{θ^α} $[\Phi_{\theta^{\alpha c}}]$ with respect to the class of sets $\mathscr{K} \ni \emptyset, \Xi$.

6. Let $\mathfrak{M} = (N_a)_{a \in W}$, $\mathfrak{M}^c = (N_a^c)_{a \in W}$ be tables of rigid bases. For the operations $\Phi_{\theta\mathfrak{M}}$, $\Phi_{\theta\mathfrak{M}^c}$, we introduce thinning conditions. We shall say that a collection of tuples η_x satisfies condition a_1^ν $[a_3^\nu]$ if in the branching table $\langle \eta_x(T_W^\nu), < \rangle$ there is a node $((a_i))_{i \in J'}$ from which one can form a $> \aleph_\nu$ $[\geqslant \aleph_\nu]$ R-(R^c-) covering of the tuple a; a_2^ν, if in the collection $\eta_x(T_W^\nu)$ there is a tuple and such an R-(R^c-) covering of it in which $\aleph_\nu$ tuples have the property of bisection; a_4^ν, if the table $\langle \mu_x, < \rangle$ attains its rank ω_ν; a_7^ν, if the collection μ_x includes a disjunctive $\aleph_\nu$-subset of tuples.

* A. D. Taimanov ([7]) gave an example of a δS-operation with a rigid base for which the complementary δS-operation has no rigid base.

Then for the operations $\Phi_{\theta_{\mathfrak{M}}}, \Phi_{\theta^c_{\mathfrak{M}}}$ the following holds:

1) for $\nu < \omega_\nu$:

$$\overline{M}_x > \mathfrak{N}_\nu \Longleftrightarrow a_1^\nu \vee a_2^\nu, \qquad \overline{M}_x \geq \mathfrak{N}_\nu \Longleftrightarrow a_2^\nu \vee a_3^\nu, \qquad \overline{M}_x = \mathfrak{N}_\nu \Longleftrightarrow \neg(a_1^\nu \vee a_2^\nu)\ \&\ a_3^\nu;$$

2) (∗∗)

$$\overline{M}_x \geq \mathfrak{N}_\nu\ \&\ \neg a_3^\nu\ \&\ \neg a_7^\nu \Rightarrow a_4^\nu;$$

3) (∗∗).

$$\overline{\overline{M}}_x \geq \mathfrak{N}_\nu \Longleftrightarrow a_3^\nu \vee a_4^\nu \vee a_7^\nu.$$

For the operation $\Phi_{\theta^c_{\mathfrak{M}}}$ the following holds:

4) (∗)

$$\overline{\overline{M}}_x > \mathfrak{N}_\nu \Longleftrightarrow a_1^\nu \vee a_2^\nu;$$

5) (∗)

$$\overline{\overline{M}}_x = \mathfrak{N}_\nu \Longleftrightarrow \neg(a_1^\nu \vee a_2^\nu)\ \&\ a_3^\nu.$$

7. Let the W-base U coincide with the base $\theta_{\mathfrak{M}}$ or $\theta^c_{\mathfrak{M}}$. For $k = 2, 4, 7$ put

$$\Phi_{H_{a_k}U}(E_a) = \{x \in \Phi_U(E_a) : \text{ the set } \eta_x \text{ satisfies the condition } a_k^\nu\}.$$

(∗) For the class of sets $\mathscr{K} \ni \emptyset,\ \Xi$, the operations

$$\Phi_{H_{a_2}\theta^\alpha}, \quad \Phi_{H_{a_2}\theta^{\alpha c}}, \quad 0 \leq \alpha < \omega_{\nu+1},$$

are weaker than the operations $\Phi_{\theta^\alpha}, \Phi_{\theta^{\alpha c}}$, respectively.

(∗∗) If conditions 1^0 and 2^0 are fulfilled, the operations

$$\Phi_{H_{a_4}\theta^\alpha_N}, \quad \Phi_{H_{a_7}\theta^\alpha_N}, \quad 2 \leq \alpha < \omega_{\nu+1},$$

are weaker than the operation $\Phi_{\theta^\alpha_N}$, and the operations

$$\Phi_{H_{a_4}\theta^{\alpha c}_N}, \quad \Phi_{H_{a_7}\theta^{\alpha c}_N}, \quad 1 \leq \alpha < \omega_{\nu+1},$$

are weaker than the operation $\Phi_{\theta^{\alpha c}_N}$.

(∗∗) If conditions 1^0 and 2^0 are fulfilled, the property $H_{\mathfrak{N}_\nu}$ is M-regular if the base M coincides with the base θ^α_N, $2 \leq \alpha < \omega_{\nu+1}$, $\theta^{\alpha c}_N$, $1 \leq \alpha < \omega_{\nu+1}$, in particular, with the base θ^α for $2 \leq \alpha < \omega_{\nu+1}$, and $\theta^{\alpha c}$ for $1 \leq \alpha < \omega_{\nu+1}$.

(∗) The property $H_{\mathfrak{Z}_\nu}$ is $\theta^{\alpha c}$-regular for the class of sets $\mathscr{K} \ni \emptyset, \Xi$.

8. Let $J = \{\alpha : \alpha < \omega_\nu\}$. If each chain ξ of the rigid base M is ordered in the order of increase of its elements, then we obtain a rigid reduced base $\breve{M}$. Let $\breve{M} \subset J^{\omega_\nu}$. Put

$$\Phi_{H_c\breve{M}}(E_i) = \{x \in \Phi_M(E_i) : \text{ the closure of the set } \breve{M}_x \text{ is not } \mathfrak{N}_\nu\text{-bicompact}\},$$

where $\breve{M}_x$ is the set of all chains of the base $\breve{M}$ whose kernels contain the point x. Let

$$q = (i_\beta)_{\beta<\alpha} \in W \quad \text{and} \quad (\beta' < \beta'') \Rightarrow (i_{\beta'} < i_{\beta''}).$$

For a system of sets (E_i) put $E_i^q = \emptyset$, if

$$(\forall \beta < \alpha)(i \neq i_\beta) \ \& \ (\exists \beta_0)[i < i_{\beta_0}],$$

and $E_i^q = E_i$ in all remaining cases. Let

$$\mathcal{E}_q = \Phi_M(E_i^q), \qquad S'_\alpha = \bigcup_{q \in W} \mathcal{E}_{qp}.$$

Then

$$(*) \qquad \Phi_{H_c\breve{M}}(E_i) = \bigcup_{q \in W} \overline{\lim_i}^{\mathfrak{N}_\nu} S_{qi}.$$

$(*)$ If the rigid base M is $\mathfrak{N}_\nu$-proper for the class of sets $\mathcal{K} \ni \emptyset$, then the property H_c is M-regular.

$(*)$ The property H_c is M-regular for the class of sets $\mathcal{K} \ni \emptyset, \Xi$, if the base M coincides with the base $\chi_\alpha, \chi_\alpha^c,\ 0 \leq \alpha < \omega_{\nu+1}$, and also, when conditions 1^0 and 2^0 are fulfilled, with the base $\theta_N^\alpha, \theta_N^{\alpha c},\ 1 \leq \alpha < \omega_{\nu+1}$, in particular, with the base $\theta^\alpha, \theta^{\alpha c},\ 0 \leq \alpha < \omega_{\nu+1}$.

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Note: Figure translations are in progress. See original paper for figures.