ON THE THEORY OF BIFURCATIONS OF POINT TRANSFORMATIONS

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Abstract

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Bifurcations of one-dimensional point transformations. Among the possible changes in the topological structure of a point transformation there may occur bifurcations of the existence of multiple cycles ([1]) and bifurcations of their stability. Bifurcations occurring inside the domain of definition of the transformation and on its boundary will be called, respectively, internal and external. Obviously, stability bifurcations are, in the general case, internal. For a one-dimensional point transformation there exist two types of stability bifurcations of a multiple cycle ([1]), corresponding to the characteristic root of the cycle passing through $+1$ and through -1.

As is not difficult to see, in both cases a bifurcation of existence also occurs when the stability changes. Indeed, suppose that for some value of the parameters the characteristic root of an n-fold fixed point is $\lambda_n = +1$. Obviously, what then occurs is the coalescence of two n-fold cycles of different stability, as a result of which both cycles disappear. When the parameters are changed in the opposite direction, a semistable n-fold cycle is born, which then splits into two n-fold cycles—one stable and one unstable. We shall call such a bifurcation natural and denote the corresponding bifurcation boundary by B_{+n}.

Suppose now that the characteristic root of an n-fold cycle is $\lambda_n = -1$. In this case there occurs the coalescence of a cycle of multiplicity n with a cycle of multiplicity $2n$, as a result of which the first changes stability and the second disappears. Under the reverse change of the parameter, from a stable (unstable) n-fold cycle there is born a stable (unstable) cycle of doubled multiplicity, while the n-fold cycle itself changes stability. We shall call such a bifurcation even and denote the corresponding boundary by B_{-n}.

Natural and even bifurcations exhaust the possible types of internal bifurcations of one-dimensional point transformations. It follows from this that cycles of even multiplicities may be born as a result of bifurcations of both types, whereas cycles of odd multiplicities can arise only by natural bifurcations.

The method of parametric diagrams. Let a one-dimensional point transformation T be defined by a sequence function $f(x,h)$. Without loss of generality, it may be assumed that it depends on one parameter h. Fixed points of multiplicity n are found as the roots of the equation

$$\Psi_n(x,h) \equiv f_n(x,h) - x = 0, \tag{1}$$

where $f_n(x,h)$ is the n-th iteration of the sequence function $f(x)$. The function $\Psi_n(x,h) = 0$ and equation (1) will be called, respectively, the function and the equation of n-fold fixed points. Let E_n denote the set of divisors of the number n, excluding n itself. Clearly, the roots of the equation $f_i(x,h) - x = 0$ are also roots of equation (1) if $i \in E_n$.

In the study of concrete point mappings, the equation of fixed points (1) is often not solvable with respect to the coordinate x, which makes it difficult to determine the number of fixed points and their dependence on the parameters. However, in many cases the equation of fixed points can be solved with respect to one parameter or another. In other cases this can be achieved by a linear change of variables and parameters.

In this connection it is expedient to study a point transformation by means of the parametric diagram (2), which is the totality of the curves (1) in the plane (x,h). Such a study is especially simple in the case when equations (1) are solved with respect to the parameter h in the form $h = F_n(x)$. It is clear that each function $\Psi_n(x,h) = 0$ has branches coinciding with the branches of the functions $\Psi_i(x,h) = 0$, if $i \in E_n$. Therefore, in what follows, by the function $\Psi_n(x,h) = 0$ we shall mean only its branches that determine n-fold fixed points. On the basis of Theorem 4 in ([1]), the point transformation T has no multiple fixed points if and only if, for any x, $F_2(x) = F_1(x)$. The maximum number of simple fixed points is greater by one than the number of extrema of the continuous function $h = F_1(x)$.

The extremal values of the functions of fixed points (1) correspond to internal bifurcations of the point transformation. An even bifurcation (the boundary B_{-n}) corresponds to a point of intersection of the curves $\Psi_n(x,h) = 0$ and $\Psi_{2n}(x,h) = 0$, and this point is an extremum of the function $\Psi_{2n}(x,h) = 0$. The remaining extrema of the functions (1) correspond to natural bifurcations (B_{+n}). Let us also note that, for any i and $k > i$, the functions $\Psi_i(x,h) = 0$ and $\Psi_k(x,h) = 0$ have no common points if $k \neq 2i$. Each function $\Psi_n(x,h) = 0$ divides the plane (x,h) into two regions: in one of them $f_n(x) > x$, and in the other $f_n(x) < x$. This circumstance also makes it possible to determine the stability character of the corresponding fixed points.

Thus the parametric diagram makes it possible to carry out a complete study of a point transformation: to find all fixed points, to investigate their stability and their dependence on the parameters, and to determine the bifurcation boundaries and the basins of attraction of stable fixed points. Nevertheless, the study of concrete point transformations often turns out to be a very complicated problem. This leads to the necessity of using, in each concrete problem, special techniques and methods peculiar only to that problem. However, any continuous sequence function can be approximated by a polynomial of sufficiently high degree so that the qualitative structure of the original transformation does not change ([3]).

In this connection, the study of point transformations with a polynomial sequence function is of interest. In principle, in such a study the topological structure of the transformation and its dependence on the coefficients of the polynomial can be completely determined. This may make it possible, generally speaking, to reduce the problem of qualitative integration of any system of second-order differential equations to the construction of a point transformation and its approximation by a power polynomial. Moreover, it is clear that the polynomial point transformation is also of independent interest.

Polynomial point transformation and its bifurcations. Let a point transformation T be given with the sequence function:

$$f(x) = a_0 x^N + a_1 x^{N-1} + a_2 x^{N-2} + \dots + a_{N-1} x + a_N. \tag{2}$$

Let us consider the question: under what conditions imposed on the coefficients of the polynomial $f(x)$ does the point transformation T have only simple fixed points? On the basis of Theorem 4 in ([1]), this question reduces to the study-

to the existence of double cycles. The polynomial $\Psi_2(x) = f[f(x)] - x$ can be represented as the product $\Psi_2(x) = q_1(x) \cdot q_2(x)$.

The function $q_2(x) = b_0 x^m + b_1 x^{m-1} + \dots + b_m$ is a polynomial of degree $m = N(N-1)$, each root of which corresponds to a double fixed point of the transformation T.

Consequently, a point transformation T with a polynomial iteration function has no multiple cycles if and only if the polynomial

$$q_2(x) = \{f[f(x)] - x\}/\{f(x) - x\}$$

has no real roots.

Let us find, in the space of coefficients of the polynomial $f(x)$, the equation of the bifurcation surface bounding the domain of existence of only simple fixed points. This equation can be obtained by eliminating the variable x from the system

$$q_2(x) = 0, \quad dq_2(x)/dx = 0. \tag{3}$$

As is known, two polynomials have common roots if their resultant is equal to zero. Consequently, the equation of the bifurcation surface for the birth of double cycles has the form

$$R[q_2(x),\, dq_2(x)/dx] = 0. \tag{4}$$

Equation (4) embraces both the natural (B_{+2}) and the even (B_{-1}) type of bifurcations. If, however, it is known from some indirect indications that the birth of double cycles can occur only by means of an even bifurcation, then the equation of the corresponding bifurcation surface (B_{-1}) has a simpler form and is expressed directly in terms of the coefficients of the polynomial $f(x)$ as

$$R\left[\Psi_1(x),\ \frac{df(x)}{dx}+1\right] \equiv \begin{vmatrix} a_0 & a_1 & a_2 \dots a_N & 0 & 0 & \dots & 0 \\ 0 & a_0 & a_1 \dots a_{N-1} & a_N & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & & & \\ 0 & 0 & 0 \dots a_0 & a_1 & a_2 & \dots & a_N \\ Na_0 & (N-1)a_1 & \dots\dots\dots & (a_{N-1}+1) & 0 & \dots & 0 \\ 0 & Na_0 & (N-1)a_1 & \dots\dots\dots & (a_{N-1}+1) & \dots & 0 \\ \vdots & \vdots & & & & & \\ 0 & 0 & \dots\dots & Na_0 & (N-1)a_1 & \dots\dots\dots & (a_{N-1}+1) \end{vmatrix} = 0 \quad (5)$$

In an analogous way, the equations of the bifurcation surfaces B_{+n}, B_{-n} for fixed points of higher multiplicities may be determined. It is clear that each function $\Psi_n(x)$ in equation (1), which in the case under consideration is a polynomial of degree N^n, can be represented as a product of polynomials, with the number of factors equal to the number of elements of the set E_n.

For a given N, construct the sequence of numbers:

$$e_1 = N, \quad e_2 = N^2 - e_1, \quad e_3 = N^3 - e_1, \quad e_4 = N^4 - e_2 - e_1, \dots$$
$$\dots,\ e_n = N^n - \sum_i e_i,$$

where i assumes all values in E_n.

Similarly, construct the sequence of functions:

$$q_1(x) = \Psi_1(x), \quad q_2 = \Psi_2(x)/q_1(x), \quad q_3(x) = \Psi_3(x)/q_1(x):$$
$$q_4(x) = \Psi_4(x)/q_1(x)\cdot q_2(x), \dots, \quad q_n(x) = \Psi_n(x)\left[\prod_i q_i(x)\right]^{-1},$$

where i assumes all values in E_n.

Each function $q_n(x)$ is a polynomial of degree e_n, each root of which corresponds to a fixed point of multiplicity n. Obviously, the number e_n is always divisible by n, and the ratio e_n/n is equal to the maximum number of cycles of multiplicity n.

As was shown above, the internal bifurcations of n-fold cycles are determined by the systems

$$q_n(x) = 0, \qquad df_n(x)/dx \mp 1 = 0, \qquad (6)$$

where the plus and minus signs are taken, respectively, for the boundaries B_{+n} and B_{-n}. The equations of the bifurcation surfaces B_{+n} and B_{-n} have, respectively, the form

$$R[q_n(x),\, df_n(x)/dx \mp 1] = 0. \quad (7)$$

It should be noted that the equations of the bifurcation surfaces B_{+n} can also be written in the simpler form

$$R[q_n(x),\, dq_n(x)/dx] = 0.$$

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Note: Figure translations are in progress. See original paper for figures.