# THE MAXIMUM PRINCIPLE FOR OPTIMIZING SYSTEMS WITH AFTEREFFECT

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## Abstract

## Full Text

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*MATHEMATICS*

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# THE MAXIMUM PRINCIPLE FOR OPTIMIZING SYSTEMS WITH AFTEREFFECT

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1. Consider the motion $x(t) = \{x_1(t), \dots, x_n(t)\}$, $t \in T = [t_0, t_1]$, specified by the equations

$$dx_i(t)/dt = X_i(x(t,\cdot), u(t,\cdot), t), \qquad i = 1, \dots, n, \tag{1}$$

$$x(t) = \begin{cases} x_0, & t = t_0, \\ \Phi_1(t), & t_0 - h \le t < t_0, \end{cases} \qquad u(t) = \{\Phi_2(t),\, t_0 - h \le t \le t_0\},$$

where $x(t,\cdot) = \{x(\tau),\, t - h \le \tau < t\}$; $u(t,\cdot) = \{u_1(\tau), \dots, u_r(\tau),\, t - h \le \tau < t\}$, $h \ge 0$; $X_i$ are functionals defined on continuous $x(t,\cdot)$ and piecewise-continuous $u(t,\cdot)$ functions.

As the class of admissible controls we take the set of piecewise-continuous functions $u(t)$ with values in a constrained set:

$$u(t) \in U, \quad t \in T.$$

Among the admissible controls it is required to find an optimal $u^0(t)$ (see (1)):

$$I(u^0) \le I(u) = \varphi(x(t_1)).$$

2. Let $u(t)$, $\tilde{u}(t) = u(t) + \Delta u(t)$, $t \in T$, be admissible controls; $x(t)$, $\tilde{x}(t) = x(t) + \Delta x(t)$, $t \in T$, the corresponding motions by virtue of (1). Introduce auxiliary continuous functions $\psi(t) = \{\psi_1(t), \dots, \psi_n(t)\}$, $t \in T$, $\psi(t) \equiv 0$, $t > t_1$. From the identity *

$$\psi_i(t_1)\Delta x_i(t_1) - \psi_i(t_0)\Delta x_i(t_0) = \int_{t_0}^{t_1} \psi_i(t)\Delta \dot{x}_i(t)\, dt + \int_{t_0}^{t_1} \dot{\psi}_i(t)\Delta x_i(t)\, dt$$

taking into account $\Delta x(t_0) = 0$ and the definition

$$\psi_i(t_1) = -\partial\varphi(x(t_1))/\partial x_i$$

we obtain the increment formula

$$\Delta I(u) = I(\tilde{u}) - I(u) = -\int_{t_0}^{t_1} \psi_i(t)\Delta\dot{x}_i(t)\,dt - \int_{t_0}^{t_1} \dot{\psi}_i(t)\Delta x_i(t)\,dt + O(\|\Delta x(t_1)\|). \tag{2}$$

Put

$$\pi(x, \psi, u) = \int_{t_0}^{t_1} \psi_i(t) X_i(x(t, \cdot), u(t, \cdot), t)\,dt. \tag{3}$$

* Everywhere below the symbol $x_i y_i$ denotes $\sum_{i=1}^{n} x_i y_i$, i.e., summation over repeated indices is performed.

Then from (2), (3) it follows that

$$\Delta I(u) = -[\pi(x, \psi, \tilde{u}) - \pi(x, \psi, u)] - \int_{t_0}^{t_1} \dot{\psi}_i(t)\Delta x_i(t)\,dt + O(\|\Delta x(t_1)\|) =$$

$$= -[\pi(x, \psi, \tilde{u}) - \pi(x, \psi, u)] - \delta\pi(x, \psi, \tilde{u}) - \int_{t_0}^{t_1} \dot{\psi}_i(t)\Delta x_i(t)\,dt +$$

$$+ O(\|\Delta x(t_1)\|) - \int_{t_0}^{t_1} O_1(\|\Delta x(t, \cdot)\|)\,dt.$$

Let $X_i(x(t, \cdot), u(t, \cdot), t)$ be continuous with respect to $x(t, \cdot)$ in the metric $C$. Then, using the normal form (2) of the variation $\delta\pi(x, \psi, u)$ and the definition

$$\dot{\psi}_i(t) = -\delta\pi(x, \psi, u)/\delta x_i(t), \qquad t \in T,$$

we arrive at the expression

$$\Delta I(u) = -[\pi(x, \psi, \tilde{u}) - \pi(x, \psi, u)] + \eta, \tag{4}$$

$$\eta = -\int_{t_0}^{t_1} \left[ \frac{\delta\pi(x,\psi,\tilde{u})}{\delta x_i(t)} - \frac{\delta\pi(x,\psi,u)}{\delta x_i(t)} \right] \Delta x_i(t)\, dt +$$

$$+O(\|\Delta x(t_1)\|) + \int_{t_0}^{t_1} O_1(\|\Delta x(t,\cdot)\|)\, dt.$$

Let us note that equations (1), through the functional (3), can be represented in the form

$$\dot{x}_i(t) = \delta\pi(x,\psi,u)/\delta\psi_i(t), \qquad t \in T.$$

**3.** Suppose there exists $\varepsilon_0 < +\infty$ such that, for all $\varepsilon,\ 0 < \varepsilon \le \varepsilon_0$, under the needle variation

$$\Delta u(t) = \begin{cases} 0, & t \notin [\theta, \theta+\varepsilon),\ \theta+\varepsilon \in T, \\ v, & t \in [\theta, \theta+\varepsilon),\ v \in U, \end{cases}$$

the relations

$$\|\Delta x(t)\| \le k\varepsilon, \qquad t \in T, \qquad \int_{t_0}^{t_1} \left\| \frac{\delta\pi(x,\psi,\tilde{u})}{\delta x(t)} - \frac{\delta\pi(x,\psi,u)}{\delta x(t)} \right\| dt \le k_1\varepsilon, \tag{5}$$

$$k,\ k_1 < +\infty$$

hold.

For the needle variation define the symbol $\delta_v\pi(x,\psi,u)/\delta u(t)$:

$$\pi(x,\psi,\tilde{u}) - \pi(x,\psi,u) = \varepsilon\, \delta_v\pi(x,\psi,u)/\delta u(t) + O_2(\varepsilon).$$

**Theorem 1.** Let $u^0(t),\ t \in T$, be an optimal control, and let $x^0(t),\ \psi^0(t)$ be the corresponding solutions of the equations

$$\dot{x}_i(t) = \frac{\delta\pi(x,\psi,u)}{\delta\psi_i(t)}, \qquad \dot{\psi}_i(t) = -\delta\pi(x,\psi,u)/\delta x_i(t), \qquad t \in (t_0, t_1) \tag{6}$$

with the boundary conditions specified above. Then for $u^0(t)$ the maximum condition is satisfied

$$\delta_v\pi(x^0,\psi^0,u^0)/\delta u(t) \le 0, \qquad t \in T, \qquad v \in U. \tag{7}$$

**4.** Replace the class of admissible controls: a control $u(t),\ t \in T$, is admissible if it is summable on $T$ and belongs to a convex bounded family of functions $U(\cdot)$. Suppose that, for variations of the form

$$\Delta u(t) = \tilde{u}(t) - u(t) = \varepsilon(v(t) - u(t)), \qquad 0 \le \varepsilon \le 1,$$

$$u(\cdot),\ v(\cdot) \in U(\cdot),$$

for $0 \le \varepsilon \le \varepsilon_0$, (5) is satisfied. Then from (4) it follows

**Theorem 2.** If $u^0(t)$, $t \in T$, is an optimal control; $x^0(t)$, $\psi^0(t)$ are the corresponding solutions of equations (6), then for $u^0(t)$ the condition

$$\int_{t_0}^{t_1} \frac{\delta\pi(x^0, \psi^0, u^0)}{\delta u_\nu(t)} (\nu_\nu(t) - u^0_\nu(t))\, dt \le 0 \tag{8}$$

is fulfilled for all $\nu(\cdot) \in U(\cdot)$.

If $u(\cdot)$ is an interior element of the set $U(\cdot)$, then ($U(\cdot)$ may be nonconvex)

$$\delta\pi(x^0, \psi^0, u^0)/\delta u_\nu(t) \equiv 0, \qquad t \in T. \tag{9}$$

5. Let the family $U(\cdot)$ consist of piecewise-continuous functions with values in a convex bounded set $U$. Then from (8) one obtains

**Theorem 3.** Under the conditions of Theorem 2 the inequality

$$\frac{\delta\pi(x^0, \psi^0, u^0)}{\delta u_\nu(t)} (\nu_\nu - u^0_\nu(t)) \le 0 \tag{10}$$

holds for all $\nu \in U,\ t \in T$.

**Corollary.** If on $\sigma \subset T$ the control $u^0(t) \in \operatorname{int} U$ ($U$ not necessarily convex), then on $\sigma$ (9) is fulfilled.

6. Suppose that $X_i$ are continuous with respect to $x(t,\cdot)$ in the metric $C^{(l)}$. If the first inequality in (5) is understood in the metric $C^{(l)}$, then Theorem 1 remains valid with the natural change of the boundary conditions for $x(t)$, $\psi(t)$. They change owing to the appearance of new terms arising when the variation $\delta\pi$ is reduced to the normal form (2). The corresponding calculations are easily carried out in each concrete case.

In a similar way one can prove the maximum principle (and other necessary optimality conditions) for systems described by equations

$$R^m_{ij}(p) x_j(t) = X_i(x(t,\cdot), u(t,\cdot), t), \qquad j = 1, \dots, n,$$

with the corresponding initial conditions. Here $R_{ij}^{m}(p)$ are linear differential operators of order $m$.

Introducing the adjoint operators $\overline{R}_{ij}^{m}(p)$, the equations for the basic and adjoint variables can, with the aid of (3), be represented in the form

$$R_{ij}^{m}(p)x_j(t) = \delta\pi(x,\psi,u)/\delta\psi_i(t), \qquad \overline{R}_{ij}^{m}(p)\psi_j(t) = \delta\pi(x,\psi,u)/\delta x_i(t), \quad (11)$$

which is more symmetric than (6). The form of the maximum condition (7) does not change.

Continuing the generalization, one can consider systems

$$X_i(x(t,\cdot), u(t,\cdot), t) = 0, \quad (12)$$

defined by means of functionals continuous with respect to $x(t,\cdot)$ on $C^{(l)}$. In this case the functional (3) allows one to write the equations for $x(t)$, $\psi(t)$ and the maximum condition in the form

$$\delta\pi(x,\psi,u)/\delta\psi_i(t) = 0, \qquad \delta\pi(x,\psi,u)/\delta x_i(t) = 0,$$

$$\delta_\nu\pi(x,\psi,u)/\delta u(t) \leq 0, \qquad t \in T, \quad \nu \in U, \quad (13)$$

which is the most compact and easy to remember.

7. The maximum condition (7) remains valid in the case where $X_i$ are continuous with respect to $u(t,\cdot)$ in the class $C^{(k)}$ and the set $U(\cdot)$ of admissible controls consists of functions of the same class.
8. **Remarks.** Since the problem under consideration is very general, the conditions under which the operations performed here are legitimate are not written out in detail. For concrete realizations of the problem these con—

questions are solved trivially. It is assumed throughout that the functionals $X_i$ are such that, for admissible controls, equations (6), (11), (13) have a unique solution defined on $T$. Such a requirement is natural for all works on the theory of optimal processes.

The principal conditions in the optimization problems under consideration are properties of the type (5), characterizing continuity of solutions with respect to the parameter and with respect to perturbations small on the average. It can be shown that the maximum conditions (7), (10), generally speaking, are violated if the inequalities (5) are not satisfied.

9. System (1) is a generalization of a broad class of systems encountered in applications and considered in the theory of optimal processes. In particular, equations (1) include systems of the form

$$\dot{x}_i(t) = f_i(x(t), u(t), t); \tag{14}$$

$$\dot{x}_i(t) = \sum_{p=1}^{m} f_{ip}(x(t-h_p), u(t), t); \tag{15}$$

$$\dot{x}_i(t) = \sum_{p=1}^{m} f_{ip}(x(t-h_p), \dot{x}(t-h_p), u(t), t), \qquad h_p = h_p(t), \tag{16}$$

$$h_p = h_p(t, x, u),$$

$$\dot{x}_i(t) = \int_\alpha^\beta \int_\gamma^\delta f_i(x(t), x(t-s), u(t-\sigma), t, \sigma, s)\, d\sigma\, ds \tag{17}$$

and others.

The elegance of the formulation of Pontryagin' s maximum principle for system (14) is explained largely by the use of the Hamiltonian

$$H(x, \psi, u, t) = \psi_i f_i(x, u, t).$$

However, in passing to more complicated systems (15)–(17), constructing $H$ according to the old rule, because of the cumbersome nature of the equations for $\psi$ and the cumbersome nature of the maximum condition, is not always expedient. The introduction of the functional (3) simplifies the formulation of the maximum principle (see (6), (7)) to the level of an original result [1].

The maximum principle in the form (6), (7) for a number of concrete systems of type (15)–(17) was proved in [3,4].

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*Note: Figure translations are in progress. See original paper for figures.*