

# SOME CASES OF REALIZATION OF ELEMENTS OF HOMOTOPY GROUPS OF HOMOGENEOUS SPACES BY TOTALLY GEODESIC SPHERES

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**Abstract**

**Full Text**

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**A. T. FOMENKO**

**SOME CASES OF REALIZATION OF ELEMENTS OF HOMOTOPY GROUPS OF HOMOGENEOUS SPACES BY TOTALLY GEODESIC SPHERES**

*(Presented by Academician P. S. Aleksandrov on 11 VI 1969)*

1. The existence in Riemannian manifolds of compact sets minimizing the Hausdorff measure among compact sets with prescribed homological properties was proved in a series of works by Reifenberg, Morrey, Almgren (see, for example, (5-7)) and others. However, an extremely broad formulation of the problem leads to the appearance, as solutions, of compact sets whose global structure is very far from that of manifolds. On the other hand, numerous works by other authors, for example Hsiang, Wolf, Helgason (see (2, 8, 9)), while establishing the existence of sufficiently broad and interesting classes of minimal (and even totally geodesic) submanifolds, leave aside the question of their homological or homotopic characteristic with respect to the ambient manifold. Moreover, the isoclinic spheres studied by Wolf (see (2)) in Grassmann manifolds are poorly suited for realizing elements of homotopy groups, since any isoclinic sphere of nonmaximal dimension is contained in a maximal isoclinic sphere and therefore is contractible to a point. In the present note an attempt is made, in some cases, to combine the advantages of these two directions.
2. Quite many examples can be given of totally geodesic submanifolds realizing cycles. Among them, for example, are the real projective spaces  $RP^{2k+1}$  in  $SO(n)$ , representing a basis of rational cycles and discovered already by L. S. Pontryagin. A more general case is given by the totally geodesic submanifolds  $RP^{n-1}$  in  $SO(n)$ , realizing the characteristic homotopy classes for the standard fibrations

$$SO(n) \rightarrow SO(n+1) \rightarrow S^n.$$

We note that the corresponding cycles and characteristic classes for the standard fibrations of the groups  $SU(n)$  and  $Sp(n)$  are no longer submanifolds.

Let us consider the group  $U(n)$  as a submanifold of the sphere  $S^{2n^2-1}$  of radius  $\sqrt{n}$  in the Euclidean space of complex matrices ( $n \times n$ ) with scalar product

$$\varphi(A, B) = \operatorname{Re} \operatorname{Sp} AB^*,$$

where  $B^* = \overline{B}^T$ . A sphere  $S^p \subset U(n) \subset S^{2n^2-1}$  will be called a central section if  $S^p = \Pi_{p+1} \cap S^{2n^2-1}$ , where  $\Pi_{p+1}$  is a  $(p+1)$ -dimensional plane passing through the origin (real dimension). We shall similarly consider central sections in the groups  $SU(n)$  and  $\operatorname{Sp}(n)$ . It is easy to see that the dimension of a central section cannot exceed the number  $s(n)$  for  $SO(n)$ ,  $s(2n)$  for  $U(n)$ , and  $s(4n)$  for  $\operatorname{Sp}(n)$ , where  $s(r)$  is the maximal number of linearly independent vector fields on the sphere  $S^{r-1}$ .

**3. Theorem 1.** Let  $m$  and  $k$  be integers,  $0 \leq m \leq k$ ,  $s = 2(k - m) + 1$ . Then in the homotopy groups  $\pi_s(X)$  listed below, the following elements are realized by totally geodesic spheres  $S^s$ :

I.  $X = SU(n)$ ,  $n \geq 2^k \geq 2$ ,  $k - m \geq 1$ ;  $\{1, 2, 3, 4, \dots, 2^m\} \in \mathbb{Z} = \pi_s(X)$ . If  $n = 2^k$ , then the element  $2^m$  is realized by a central section.

II.  $X = SO(n)$ ,  $n \geq 2^{k+1}$ ,  $k - m \geq 3$ ;

a)  $s \equiv 3 \pmod{8}$ ,  $\{1, 2, 3, 4, \dots, 2^m\} \in \mathbf{Z} = \pi_s(X)$ ;

b)  $s \equiv 7 \pmod{8}$ ,  $\{2, 4, 6, 8, \dots, 2^{m+1}\} \in \mathbf{Z} = \pi_s(X)$ ;

c)  $s = (2k + 1) \equiv 1 \pmod{8}$ ,  $\{1\} \in \mathbf{Z}_2 = \pi_s(X)$ .

If  $n = 2^k$ , then the elements  $2^m$ ,  $2^{m+1}$ ,  $1$  in cases a), b), c), respectively, are realized by central sections.

III.  $X = \operatorname{Sp}(n)$ ,  $n \geq 2^k$ ,  $k - m \geq 5$ ;

a)  $s \equiv 7 \pmod{8}$ ,  $\{1, 2, 3, 4, \dots, 2^m\} \in \mathbf{Z} = \pi_s(X)$ ;

b)  $s \equiv 3 \pmod{8}$ ,  $\{2, 4, 6, 8, \dots, 2^{m+1}\} \in \mathbf{Z} = \pi_s(X)$ ;

c)  $s = (2k + 1) \equiv 5 \pmod{8}$ ,  $\{1\} \in \mathbf{Z}_2 = \pi_s(X)$ .

If  $n = 2^k$ , then the elements  $2^m$ ,  $2^{m+1}$ ,  $1$  in cases a), b), c), respectively, are realized by central sections.

**Remark 1.** It is clear that in Theorem 1 the dimensions of the central sections more or less accurately approximate from below the number  $s(r)$ —the greatest possible dimension for them, while the insignificant deviation from  $s(r)$  depends on  $k \pmod{8}$ .

**Theorem 2.** Let  $m$  and  $k$  be integers,  $0 \leq m \leq k$ . Then, in the homotopy groups  $\pi_s(X)$  listed below, the following elements are realized by totally geodesic spheres:

I.  $X = \mathfrak{G}^C(n, t)$ —the complex Grassmann manifold,  $n = 2^k + r$ ,  
 $t = 2^{k-1} - p$ ,  $r \geq p \geq 0$ ,  $k - m \geq 2$ ,  $s = 2(k - m)$ ;  $\{1, 2^m\} \in \mathbf{Z} = \pi_s(X)$ .

If  $r = p = 0$ , then the element  $2^m$  is a central section ( $X \subset SU(2^k)$ ).

II.  $X = SO(2^{k+1})/U(2^k)$ ,  $k - m \geq 3$ ,  $s = 2(k - m)$ ;

a)  $s \equiv 2 \pmod{8}$ ,  $\{2^m\} \in \mathbf{Z} = \pi_s(X)$ ;

b)  $s \equiv 6 \pmod{8}$ ,  $\{2^{m+1}\} \in \mathbf{Z} = \pi_s(X)$ ;

c)  $s = 2k \equiv 0 \pmod{8}$ ,  $\{1\} \in \mathbf{Z}_2 = \pi_s(X)$ .

III.  $X = U(2^{k+1})/Sp(2^k)$ ,  $k - m \geq 4$ ,  $s = 2(k - m) + 1$ ;

a)  $s \equiv 1 \pmod{8}$ ,  $\{2^m\} \in \mathbf{Z} = \pi_s(X)$ ;

b)  $s \equiv 5 \pmod{8}$ ,  $\{2^{m+1}\} \in \mathbf{Z} = \pi_s(X)$ ;

c)  $s = (2k + 1) \equiv 7 \pmod{8}$ ,  $\{1\} \in \mathbf{Z}_2 = \pi_s(X)$ .

IV.  $X = \mathfrak{G}^H(2^k, 2^{k-1})$ —the quaternionic Grassmann manifold,  $s = 2(k - m)$ ,  $k - m \geq 6$ ;

a)  $s \equiv 0 \pmod{8}$ ,  $\{2^m\} \in \mathbf{Z} = \pi_s(X)$ ;

b)  $s \equiv 4 \pmod{8}$ ,  $\{2^{m+1}\} \in \mathbf{Z} = \pi_s(X)$ ;

c)  $s = 2k \equiv 6 \pmod{8}$ ,  $\{1\} \in \mathbf{Z}_2 = \pi_s(X)$ .

V.  $X = Sp(2^k)/U(2^k)$ ,  $k - m \geq 5$ ,  $s = 2(k - m)$ ;

a)  $s \equiv 6 \pmod{8}$ ,  $\{2^m\} \in \mathbf{Z} = \pi_s(X)$ ;

b)  $s \equiv 2 \pmod{8}$ ,  $\{2^{m+1}\} \in \mathbf{Z} = \pi_s(X)$ ;

c)  $s = 2k \equiv 4 \pmod{8}$ ,  $\{1\} \in \mathbf{Z}_2 = \pi_s(X)$ .

VI.  $X = U(2^{k+1})/O(2^{k+1})$ ,  $k - m \geq 4$ ,  $s = 2(k - m) + 1$ ;

a)  $s \equiv 5 \pmod{8}$ ,  $\{2^m\} \in \mathbf{Z} = \pi_s(X)$ ;

b)  $s \equiv 1 \pmod{8}$ ,  $\{2^{m+1}\} \in \mathbf{Z} = \pi_s(X)$ ;

c)  $s = (2k + 1) \equiv 3 \pmod{8}$ ,  $\{1\} \in \mathbf{Z}_2 = \pi_s(X)$ .

VII.  $X = \mathfrak{G}^R(2^{k+1}, 2^k)$ —the real Grassmann manifold;

a)  $s \equiv 4 \pmod{8}$ ,  $\{2^m\} \in \mathbf{Z} = \pi_s(X)$ ;

b)  $s \equiv 0 \pmod{8}$ ,  $\{2^{m+1}\} \in \mathbf{Z} = \pi_s(X)$ ;

c)  $s = 2k \equiv 2 \pmod{8}$ ,  $\{1\} \in \mathbf{Z}_2 = \pi_s(X)$ .

**Remark 2.** All manifolds  $X$ , (II)–(VII), may be regarded as totally geodesic submanifolds in the group  $SO(2^{k+i})$ , where  $i$  in

in different cases takes the values 1, 2, or 3, and then all elements from (II)–(VII) are realized by central sections.

**Theorem 3.** Consider the  $\widetilde{\mathfrak{G}}_{2k+2,2}^R$ -covering over  $\mathfrak{G}_{2k+2,2}^R$ ,  $k \geq 2$ . Then the group  $\pi_{2k}(\widetilde{\mathfrak{G}}_{2k+2,2}^R)$  contains an element of infinite order realized by a totally geodesic sphere  $S^{2k}$ .

4. The proof of Theorems 1, 2, 3 is based on the study of Bott periodicity for the unitary and orthogonal groups. It is known (see, for example, (1)) that the generator of the group  $\pi_{2n-1}(GL(2^{n-1}, C)) = Z$  contains the mapping  $f_{2n-1}$ , which we shall now describe. Let  $f$  and  $g$  be two continuous mappings  $f : S^{n-1} \rightarrow GL(N, C)$ ,  $g : S^{m-1} \rightarrow GL(M, C)$ . Extending  $f$  and  $g$  by homogeneity to  $\mathbf{R}^n$  and  $\mathbf{R}^m$ , respectively, we may consider the continuous mapping

$$\omega(x, y) = \begin{pmatrix} f(x) \otimes 1_N, & -1_M \otimes g^*(y) \\ 1_M \otimes g(y), & f^*(x) \otimes 1_N \end{pmatrix}, \quad f^* = f^T, \quad g^* = g^T,$$

defined on  $\mathbf{R}^{n+m} \setminus (0, 0)$ ;  $\omega(x, y)$  gives rise to a mapping  $(f * g) : S^{n+m-1} \rightarrow GL(2MN, C)$ . If  $f_1 : S^1 \rightarrow GL(1, C)$ ,  $f_1(z) = z$ , then  $f_{2n-1} = f_1 * f_1 * \dots * f_1$  ( $2n - 1$  times). The mappings  $f_{2n-1}$  generate mappings  $a_{2k+1} \in \pi_{2k+1}(SU(2^k))$ . If  $a_i : S^i \rightarrow SU(m)$  has already been defined, then the image of the sphere  $S^{i+2}$  in  $SU(2m)$  under the mapping  $a_{i+2}$  will be the submanifold

$$(E_m \oplus a_i^{-1}(x)) \cdot p \cdot (E_m \oplus a_i(x)), \quad x \in S^i,$$

$$\begin{pmatrix} \alpha E_m, & \beta E_m \\ -\beta E_m, & \bar{\alpha} E_m \end{pmatrix} = p \in S^2$$

$|\alpha|^2 + \beta^2 = 1$ ,  $\beta$  real,  $E_m$  is the identity matrix of order  $m$ . As  $a_3$  one must take the identity mapping  $S^3 \rightarrow SU(2)$ . The mapping  $a_{2k+1}$  is a smooth embedding of the sphere  $S^{2k+1}$  in  $SU(2^k)$ . Since the correspondence  $a_i \rightarrow a_{i+2}$  determines the periodicity isomorphism, this thereby determines a generator  $\{a_{2k+1}\}$  in the group  $\pi_{2k+1}(SU(2^k))$ , since  $a_3 \in \{1\} \in \pi_3(SU(2))$ .

It turns out that  $a_{2k+1}(S^{2k+1})$  is a totally geodesic sphere in  $SU(2^k)$ . The minimal subgroup in the group  $SU(2^k)$  containing the sphere  $S^{2k+1}$  is the group  $\text{Spin}(2k + 2)$ , in which the sphere  $S^{2k+1}$  determines the characteristic class  $\alpha \in \pi_{2k+1}(\text{Spin}(2k + 2))$  of the standard fibration  $\text{Spin}(2k + 2) \rightarrow$

$\text{Spin}(2k+3) \rightarrow S^{2k+2}$ , and its totally geodesic equator  $S^{2k}$  is the characteristic class  $\alpha' \in \pi_{2k}(\text{Spin}(2k+1))$ . Again using Bott periodicity and the fact that

$$SU(2^k) \cap su(2^k) = \mathfrak{G}_C(2^k, 2^{k-1})$$

(here the group  $SU(2^k)$  and its Lie algebra  $su(2^k)$  are regarded as submanifolds in the space of matrices  $(2^k \times 2^k)$ ), we, interpreting  $\mathfrak{G}_{2k+2,2}^R$  as the set of symmetric orthogonal matrices of signature two (the number of negative squares), obtain Theorem 3.

The group  $SU(n)$  can be embedded in a standard way in  $SO(2n)$ , which makes it possible to obtain totally geodesic spheres in  $SO(2^{k+1})$ , and then to apply Bott periodicity for the orthogonal group, since the sphere  $S^{2k+1}$  is filled by minimal geodesics joining  $E$  with  $-E$ . Considering the sequence of loop spaces  $\Omega_k(n)$  ( $0 \leq k \leq 8$ ) occurring in the periodicity theorem, and passing each time from the sphere  $S^p$  to its equator  $S^{p-1}$ , we obtain Theorems 1 and 2.

**Remark 3.** In Theorem 2, (II)–(VII), the dimension of the representation space of the motion groups has the form  $2^k$ , but the theorem is easily formulated for any dimension  $n$ , using the embedding  $\Omega_k(n) \rightarrow \Omega_k(n+p)$  (see, for example, (3)), and writing out the corresponding restrictions on the dimension  $\pi_s(\tilde{X})$ .

5. Although in Theorem 3 the group  $\pi_{2k}(\tilde{\mathfrak{G}}_{2k+2,2}^R)$  is unstable, the proof nevertheless rests on Bott periodicity. In the essentially unstable

one must resort to other methods. Consider, for example, the group  $\pi_7(\text{Spin}(8)) \supset Z + Z$ . It is known (see <sup>4</sup>) that in the Lie algebra  $so(8)$  there acts Cartan's automorphism of triality  $h$ ,  $h^3 = E$ , cyclically permuting  $so(8)$  and the two irreducible components of its spin representation. It can be shown that in the 14-dimensional plane complementary to the Lie algebra of the exceptional group  $G_2 \subset \text{Spin}(8)$ , there are distinguished three 7-dimensional planes  $A, B, C$ , permuted by the automorphism  $h$  in cyclic order and generating over the field  $C$  the algebra of octaves without unity. It turns out that  $\exp A = S_1^7$ ,  $\exp B = S_2^7$ ,  $\exp C = S_3^7$ . It is easily seen that if  $h$  is extended to an automorphism  $H$  of the group  $\text{Spin}(8)$ , then  $H_*(\alpha) = -\beta$ ,  $H_*(\beta) = \gamma$ ,  $H_*(\gamma) = -\alpha$ , where  $\alpha, \beta, \gamma$  are the elements of  $\pi_7(\text{Spin}(8))$  corresponding to the spheres  $S_1^7, S_2^7, S_3^7$ , respectively, and  $H_*$  is the induced automorphism,  $H_*^3 = 1$ . It can be verified that  $\gamma$  is the characteristic class of the bundle

$$\text{Spin}(8) \xrightarrow{i} \text{Spin}(9) \rightarrow S^8,$$

$\alpha = \beta + \gamma$ , and no two of these three elements are proportional.

**Theorem 4.** In the group  $\pi_7(\text{Spin}(8))$ , the elements of infinite order  $\alpha, \beta, \gamma$  are realized by totally geodesic spheres  $S^7$ . In the group  $\pi_7(SO(8))$ , the elements  $\alpha$  and  $\beta$  are realized by central sections.

**Remark 4.** Since  $i_*(\gamma) = 0$ , in  $\pi_7(SO(n))$  for  $n \geq 9$  we obtain the element  $i_*(\beta) = i_*(\alpha) \neq 0$  of infinite order, realized by a totally geodesic sphere  $S^7$ .

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Faculty of Mechanics and Mathematics of Moscow State University named after M. V. Lomonosov

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*Note: Figure translations are in progress. See original paper for figures.*

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