ON STABILITY IN THE THREE-BODY PROBLEM

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ON STABILITY IN THE THREE-BODY PROBLEM

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A generalization of Lyapunov' s second method, proposed in works (^{1,2}), makes it possible to carry out a stability investigation in the three-body problem.

We shall write the equations of motion in the three-body problem in the canonical variables L, λ, ρ, ω (^{3,4})

$$\frac{d\lambda}{dt} = \frac{\partial F}{\partial L}, \qquad \frac{dL}{dt} = -\frac{\partial F}{\partial \lambda},$$

$$\frac{d\omega}{dt} = \frac{\partial F}{\partial \rho}, \qquad \frac{d\rho}{dt} = -\frac{\partial F}{\partial \omega}, \tag{1}$$

where $L = m\sqrt{M} \cdot \sqrt{a}$ (a is the semimajor axis, m the mass of the planet, M the mass of the Sun), $\rho_1 = L(1 - \sqrt{1 - e^2})$ (e is the eccentricity), $\rho_2 = (L - \rho_1)(1 - \cos i)$ (i is the inclination), λ is the mean longitude, ω_1 is the longitude of perihelion, ω_2 is the longitude of the node. The variables $L_1, \lambda_1, \rho_1, \omega_1, \rho_2, \omega_2$ refer to one planet, $L_2, \lambda_2, \rho_3, \omega_3, \rho_4, \omega_4$ to the other. $F = F_0 + \mu F_1$; $F_0 = -M_1(2L_1^2)^{-1} - M_2(2L_2^2)^{-1}$, where M_1 and M_2 are constants; μF_1 is the perturbing function, μ is a small parameter having the meaning of the ratio of the planet' s mass to the mass of the Sun; the masses of the planets are assumed to be of the same order of magnitude,

$$\mu F_1 = \sum A \rho_1^{q_1} \rho_2^{q_2} \rho_3^{q_3} \rho_4^{q_4} \cos\left(\sum k_i \lambda_i + \sum p_i \omega_i + h\right). \tag{2}$$

Here k_i and p_i are integers over which the summation is performed. A and h depend only on L, $2q_i$ are positive integers, and $2q_i \geq |p_i|$.

The properties of the perturbing function are studied in detail by H. Poincaré (^{3,4}); here we shall only write down the integrals which will be needed below,

$$\sum L - \sum \rho = \text{const},$$

$$\rho_2\left(L_1-\rho_1-\frac{\rho_2}{2}\right)=\rho_4\left(L_2-\rho_3-\frac{\rho_4}{2}\right). \tag{3}$$

We shall denote the coefficients of the series (2) by μh_{kp}, and the argument of the cos by θ. We shall also introduce the frequencies $n_1(L) = M_1 L_1^{-3}$ and $n_2(L) = M_2 L_2^{-3}$, assuming that $n_1 \neq n_2$. Equations (1) in these notations are written in the form

$$\frac{d\lambda}{dt} = n + \mu \sum_{k,p} \frac{\partial h_{kp}}{\partial L} \cos\theta,$$

$$\frac{dL}{dt} = \mu \sum_{k,p} h_{kp} k \sin\theta,$$

$$\frac{d\omega}{dt} = \mu \sum_{k,p} \frac{\partial h_{kp}}{\partial \rho} \cos\theta,$$

$$\frac{d\rho}{dt} = \mu \sum_{k,p} h_{kp} p \sin\theta. \tag{4}$$

The variables L, ρ, ω in equations (4) vary slowly—their derivatives are proportional to μ, and only the two phases λ vary rapidly,

with frequency $n + O(\mu)$. The vanishing of the combination frequencies $k_1 n_1(L) + k_2 n_2(L)$ leads to the appearance, in the right-hand sides of equations (4), of slowly varying terms, i.e., to resonance phenomena.

The lines on which the expression $k_1 n_1(L) + k_2 n_2(L)$ vanishes (resonance lines) are, in the present case, rays issuing from the origin of coordinates in the plane L,

$$L_2/L_1 = \sqrt[3]{M_2/M_1} \cdot \sqrt[3]{-k_2/k_1}. \tag{5}$$

Since the function $F = F_0 + \mu F_1$ is an integral of the motion, the integral curve in the plane L remains near the curve $F_0(L) = F_0(L_0)$.

To study stability only with respect to the variables L of a certain point L_0, we use Theorem III of [2], constructing a perturbed Lyapunov function.

Let the point L_0 lie on some resonance ray (5) for $k = k_0 = \{k_{10}, k_{20}\}$. Given $\varepsilon > 0$, we shall indicate such $\eta(\varepsilon)$, $T(\varepsilon, \mu)$, and $\mu_0(\varepsilon)$ that a solution in the variables L satisfying, at the initial moment $t = 0$, the condition $|L(0) - L_0| < \eta$, for all $0 < t < T(\varepsilon, \mu)$ and $\mu < \mu_0$ remains in an ε-neighborhood, i.e., $|L(t) - L_0| < \varepsilon$.

The integrals (3) make it possible to conclude that when L changes within $O(\varepsilon)$, the variables ρ undergo changes of the same order and, being small at the initial moment, ρ remain small for sufficiently small ε for $0 < t < T(\varepsilon, \mu)$; therefore there is no need to investigate stability with respect to the variables ρ. It is essential that series of the form (2) in small ρ will converge rapidly on this time interval of length T. From what has been said above it follows that the integrals (3) determine a certain bound, depending on the initial values of ρ, on the quantity ε from above.

Through the point L_0 there also passes the curve $F_0(L) = F_0(L_0)$. Introduce a new variable x, measured from the point L_0 along the tangent at this point to the curve $F_0(L) = F(L_0)$; the direction vector of the tangent is $\mathbf{l}$. Deviations of the integral curve along the normal to this direction in an ε-neighborhood will be small (of order $o(\varepsilon), \mu$), since $F_0 + \mu F_1$ is an integral of the motion; this permits the stability of the point L_0 to be investigated only with respect to the single variable x, $dx = (\mathbf{l} \cdot d\mathbf{L})$.

As the unperturbed Lyapunov function we choose $v_0(L) = |x|$, and we shall seek the perturbed Lyapunov function v in the form

$$v = v_0(L) + \mu v_1(L, \lambda, \rho, \omega, \varepsilon). \tag{6}$$

Differentiating v by virtue of equations (4), we obtain

$$\dot{v} = \mu \sum_{k,p} h_{kp} \cdot (kl) \sin\theta \cdot \operatorname{sgn} x + \mu \frac{\partial v_1}{\partial \lambda} \left(n + \mu \sum_{k,p} \frac{\partial h_{kp}}{\partial L} \cos\theta \right) +$$
$$+ \mu^2 \frac{\partial v_1}{\partial L} \sum_{k,p} h_{kp} k \sin\theta + \mu^2 \frac{\partial v_1}{\partial \rho} \sum_{k,p} h_{kp} p \sin\theta + \mu^2 \frac{\partial v_1}{\partial \omega} \sum_{k,p} \frac{\partial h_{kp}}{\partial \rho} \cos\theta. \tag{7}$$

From the series

$$\dot{x} \operatorname{sgn} x = \mu \sum_{k,p} h_{kp} (kl) \sin\theta \cdot \operatorname{sgn} x$$

we separate all terms whose resonance lines lie in the 2ε-neighborhood of the point L_0, and denote their sum by μR_ε. In what follows, the sum of the resonance terms will be denoted by a straight bar over the summation sign, and the sum of the oscillatory terms by a wavy one. The terms of the series belonging to k_0 shall be assigned to the oscillatory terms.

Choose some $\eta > 0$ ($\eta < \varepsilon$) and $\sigma < \frac{1}{2}(\varepsilon - \eta)$. We require that the function v_1, for $\eta < |x| < \varepsilon$, satisfy the equation

$$\frac{\partial v_1}{\partial \lambda} n = \operatorname{sgn} x \widetilde{\sum}_{k,p} h_{kp} (kl) \sin\theta. \tag{8}$$

The denominators (kn) that arise in the integration are bounded below by quantities of order η or ε. Therefore the function v_1, for $\eta < |x| < \varepsilon$, is bounded and has order $(O(\varepsilon, \eta))^{-1} \sum |h_{kp}|$. By choosing μ_0 sufficiently small one can make the perturbation μv_1 less than $\sigma/2$ for all $\mu < \mu_0$. In this case, from relations (7), taking into account equations (4), it follows that $\dot{v} = \mu R_\varepsilon + O(\mu^2)$, and for the time interval T on which a solution beginning in the η-neighborhood remains in the ε-neighborhood, according to Theorem III of paper (^{2}) the estimate is valid

$$T \sim \frac{\sigma}{2} \mu^{-1} [R_\varepsilon + O(\mu)]^{-1}.$$

Let us now estimate the magnitude of the remainder R_ε, which for fixed ε is determined by the rate of convergence of series (2). For this purpose introduce the unit vector $\chi = \left\{ \frac{k_i}{|k|} \right\}$ and consider the combination frequency $(\chi n(L)) \cdot |k|$. At the point L_0, $\chi_0 n(L_0) = 0$. On other resonance lines $(\chi n) = (\chi_0 + \Delta\chi) \cdot (n_0 + \Delta n) = 0$. On the boundary of the 2ε-neighborhood, $\Delta n = O(\varepsilon)$; consequently, for a resonance line lying within the 2ε-neighborhood, $\Delta\chi \leq O(\varepsilon)$. Thus the smallest k for which the resonance line will be in the 2ε-neighborhood is determined by the condition $k^{-1} = O(\varepsilon)$, and as ε decreases the remainder R_ε, containing all terms of this kind, decreases.

It should be noted that in the concrete three-body problem the parameter μ is a small but fixed quantity; therefore the length of the interval T cannot be increased by decreasing the parameter μ. However, this can be done by taking advantage of the smallness of R_ε. It is known (^{3}, ^{4}) that the order of a term of series (2) in powers of the small eccentricities and inclinations, belonging to $k = \{k_1, k_2\}$, is not lower than $||k_1| - |k_2||$. Suppose that $R_\varepsilon = O(\mu^2)$, and determine one more approximation for the function v. $v = v_0(L) + \mu v_1 + \mu^2 v_2$, where v_1 has already been determined above. Then

$$\begin{aligned} \dot{v} = \mu R_\varepsilon &+ \mu^2 \frac{\partial v_1}{\partial \lambda}\frac{\partial F_1}{\partial L} - \mu^2 \frac{\partial v_1}{\partial \rho}\frac{\partial F_1}{\partial \omega} + \mu^2 \frac{\partial v_1}{\partial \omega}\frac{\partial F_1}{\partial \rho} - \mu^2 \frac{\partial v_1}{\partial L}\frac{\partial F_1}{\partial \lambda} \\ &+ \mu^2 \frac{\partial v_2}{\partial \lambda}\left(n + \mu \frac{\partial F_1}{\partial L}\right) - \mu^3 \frac{\partial v_2}{\partial \rho}\frac{\partial F_1}{\partial \omega} + \mu^3 \frac{\partial v_2}{\partial \omega}\frac{\partial F_1}{\partial \rho} - \mu^3 \frac{\partial v_2}{\partial L}\frac{\partial F_1}{\partial \lambda}. \end{aligned} \tag{9}$$

We carry out here the multiplication of the series containing the derivatives of the function v_1, separate the resonance terms in the manner indicated above, and define v_2 as the solution of the equation

$$\frac{\partial v_2}{\partial \lambda} n = \frac{\widehat{\partial v_1}}{\partial L}\frac{\partial F_1}{\partial \lambda} - \frac{\widehat{\partial v_1}}{\partial \lambda}\frac{\partial F_1}{\partial L} + \frac{\widehat{\partial v_1}}{\partial \rho}\frac{\partial F_1}{\partial \omega} - \frac{\widehat{\partial v_1}}{\partial \omega}\frac{\partial F_1}{\partial \rho}. \tag{10}$$

Thus the estimate for T becomes

$$T(\varepsilon,\mu) \le \frac{\sigma(\varepsilon)}{2\left(\mu R_\varepsilon + O(\mu^3)\right)}. \tag{11}$$

The construction of higher approximations in a concrete problem may prove impossible, since ε is bounded below by the number η, which must be so large that the η-neighborhood of the resonance line includes the initial conditions of the given problem.

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Note: Figure translations are in progress. See original paper for figures.

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