ADMISSIBILITY OF THE SAMPLE MEAN AS AN ESTIMATE OF A SHIFT PARAMETER UNDER POLYNOMIAL LOSSES

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Abstract

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MATHEMATICS

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ADMISSIBILITY OF THE SAMPLE MEAN AS AN ESTIMATE OF A SHIFT PARAMETER UNDER POLYNOMIAL LOSSES

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1. We consider the problem of estimating the parameter $\theta \in R^1$ from the data of a repeated sample $(x_1, \dots, x_n)$ from a population with distribution function (d.f.) $F(x-\theta)$.

If $\tilde{\theta} = \tilde{\theta}(x_1, \dots, x_n)$ is an estimate of the parameter θ, then the losses that we incur in doing this are specified by a function $r(\tilde{\theta}; \theta)$, called the loss function (loss). The mathematical expectation of the loss function

$$R(\tilde{\theta}; \theta) = E_\theta r(\tilde{\theta}; \theta)$$

is called the risk of the estimate $\tilde{\theta}$. For a given risk, the notion of admissibility is introduced in a natural way.

If the loss is quadratic, $r(\tilde{\theta}; \theta) = (\tilde{\theta} - \theta)^2$, then for $n \geq 3$ the admissibility of the sample mean $\bar{x} = (x_1 + \dots + x_n)/n$ as an estimate of the parameter θ is equivalent to the normality of $F(x)$ ([1]) (see also ([2])).

An analogous result (under rather severe restrictions on $F(x)$) was established in ([3]) for losses

$$r(\tilde{\theta}; \theta) = \begin{cases} -\alpha(\tilde{\theta} - \theta), & \text{when } \tilde{\theta} \leq \theta, \\ \beta(\tilde{\theta} - \theta), & \text{when } \tilde{\theta} \geq \theta, \ \alpha, \beta > 0, \end{cases}$$

and losses of the form $r(\tilde{\theta}; \theta) = |\tilde{\theta} - \theta|^{2m-1}$.

Here an analogous result will be formulated that pertains to polynomial losses.

2. Let $P(u)$ be a polynomial in u of degree m with positive coefficients. Consider the loss function

$$r(\tilde{\theta}; \theta) = P((\tilde{\theta} - \theta)^2).$$

Theorem. *Let $(x_1, \dots, x_n)$ be a repeated sample from a population with d.f. $F(x-\theta)$, having two continuous derivatives and satisfying the conditions*

$$\int x\, dF(x) = 0, \tag{1}$$

$$\int x^{2m}\, dF(x) < +\infty. \tag{2}$$

Then, if the sample mean $\bar{x}$ is admissible in the class of unbiased estimates of θ, then $F(x)$ is the d.f. of the normal law.

The proof is based on the same considerations that were used in ([1,3]).

From the admissibility of $\bar{x}$ we obtain

$$\sum_{\substack{k_1+\dots+k_n=2m-1\\ k_i\geq 0}} A_{k_1\dots k_n}\varphi^{(k_1)}_{(\tau_1)}\cdots\varphi^{(k_n)}_{(\tau_n)} = 0 \tag{3}$$

for all $\tau_1, \dots, \tau_n$ connected by the relation $\sum_{i=1}^n \tau_i = 0$. Here

$$\varphi(\tau) = \int_{-\infty}^{\infty} e^{i\tau x}\, dF(x).$$

By means of a change of the unknown function and subsequent differentiation, one can verify that only two cases are possible:

1) All derivatives of the function $\varphi(x)$, evaluated at $\tau = 0$, coincide with the corresponding derivatives of the function $\exp\{Q(\tau)\}$, where $Q(\tau)$ is some polynomial. By the well-known theorem of Marcinkiewicz ([4]), this means that $\varphi(\tau)$ is the characteristic function of a normal distribution.

2) $\varphi(\tau)$ satisfies the equation

$$\varphi_\tau^{(2m-1)} = (a_1\tau + b_1)\varphi(\tau), \tag{4}$$

where a_1, b_1 are certain constants.

Applying the inverse Fourier transform to (4), we obtain that the probability density $f(x)$ has the form

$$f(x) = C\exp\{ax^{2m} + bx\}. \tag{5}$$

But the admissibility condition for $\bar{x}$, written in terms of the density, has the form

$$\int_{-\infty}^{\infty} P'(u)\prod_{i=1}^{n} f(u+z_i)\, du = 0 \tag{6}$$

for all $z_i: \sum_{i=1}^{n} z_i = 0$.

By direct substitution we verify that the density (5), for $m > 1$, cannot satisfy condition (6), i.e., case 2) is impossible.

3. Let us note that we used the differentiability of $F(x)$ only when establishing that (5) follows from (4). If $m = 1$, then (4) is integrated directly and, evidently, $\varphi(x)$ is the characteristic function of the normal law. Thus, the result of ([1]) is a special case of ours.

In conclusion, I express my deep gratitude to Yu. V. Linnik and A. M. Kagan for their attention to this work.

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Note: Figure translations are in progress. See original paper for figures.