ON UNIVERSALLY OPTIMAL CUBATURE FORMULAS

MATHEMATICS

1970

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Abstract

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UDC 518.517.392

MATHEMATICS

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ON UNIVERSALLY OPTIMAL CUBATURE FORMULAS

(Presented by Academician S. L. Sobolev on 20 VIII 1969)

1°. In this note we continue the study, begun in paper ([1]), of questions connected with the problem of optimizing algorithms for the numerical integration of functions of many variables.

Let

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \tag{1}$$

denote the column vector of coordinates of a variable point of the n-dimensional Euclidean space E_n, and let Ω be a bounded domain in this space. Following ([2]), to each cubature formula

$$\int\limits_{\Omega} \varphi(x)\, dx \simeq \sum_{k=1}^{N} c_k \varphi(x^{(k)}) \tag{2}$$

we shall associate the error functional

$$(l, \varphi) = \int\limits_{\Omega} \varphi(x)\, dx - \sum_{k=1}^{N} c_k \varphi(x^{(k)}) \tag{3}$$

and characterize the quality of the formula by the norm of this functional

$$\|e\|_{\Phi^*} = \sup \frac{|(l, \varphi)|}{\|\varphi\|_{\Phi}}. \tag{4}$$

Finding the minimum of expression (4) with respect to c_k and $x^{(k)}$ is a typical minimax problem; the c_k and $x^{(k)}$ that realize the minimax give the optimal formula. This problem is so difficult that, in its full scope, even for functions of one variable it has been solved only in a few special cases. In view of this, we shall restrict ourselves to considering its simplified variant, namely, we shall carry out the minimization only with respect to the coefficients, assuming the system of nodes to be fixed. In paper ([1]) it is shown that, in this formulation, the problem reduces to a linear problem: to determine the optimal coefficients we obtain a system of linear algebraic equations, and thus, from the theoretical point of view, the question could be regarded as exhausted. However, in practice, when computing integrals of high multiplicity, we shall have to deal with a system of very high order, and its direct solution will be quite difficult. On the other hand, the optimality of a formula strongly depends on the choice of the function space, and such a choice in itself is quite problematic, which leads to instability of conclusions concerning the optimality of various computational algorithms. In connection with this, I. Babushka ([3]) posed the problem of constructing universally optimal formulas, i.e., more or less simple formulas that are, in a certain sense, close to optimal, but at the same time do not depend on the choice of a concrete space from a given class of such spaces.

Let us give a precise definition. Let

$$h = (\operatorname{mes}\Omega/N)^{1/n}. \tag{5}$$

Denote by $l_0(\Phi, h)$ the error functional of the optimal formula. Then a formula with error functional $l_0^a(\Phi, h)$ will be called asymptotically optimal if

$$\lim_{h\to 0} \frac{\|l_0^a(\Phi, h)\|}{\|l_0(\Phi, h)\|} = 1. \tag{6}$$

Suppose we have some class of spaces $\{\Phi\}$. A formula with error functional l_0^a will be called universally asymptotically optimal in the class of spaces $\{\Phi\}$ if equality (6) holds simultaneously for all Φ in the given class of spaces.

2°. Let us first dwell on the integration of periodic functions. We shall consider the set of periodic generalized functions with fundamental period matrix T, assuming that $|T| = 1$. As shown in (2), the set of such functions (for brevity we shall call them T-periodic) is isomorphic to the set of generalized functions on the n-dimensional torus of unit volume, which is obtained by identifying all points of n-dimensional Euclidean space that differ by a period. We define the space $\widetilde{H}^{(\mu)}$ as the collection of those T-periodic generalized functions for which the sum is finite:

$$\|u\|^2 = \sum_{\gamma} |\hat{u}(\gamma)|^2 \mu^2(\gamma T^{-1}) < \infty. \tag{7}$$

Here $\hat{u}(\gamma)$ denotes the Fourier coefficients of the function $u(x)$, depending on the integer-valued n-dimensional vector γ, and μ plays the role of a weight function. Denote by $\widetilde{\mathfrak{M}}$ the class of spaces $\widetilde{H}^{(\mu)}$ whose weight functions satisfy the condition

$$\mu(\gamma T^{-1}) \geq A(a + |\gamma T^{-1}|^m), \tag{8}$$

where $A > 0$, $a \geq 0$, and $m > n/2$. The specific values of the constants are immaterial for us.

It follows from the embedding theorems that all spaces of the class $\widetilde{\mathfrak{M}}$ are embedded in the space C. We shall consider cubature formulas with a system of nodes $x^{(\beta)} = hH\beta$, forming a regular lattice. Assume also that the torus over which the integral is computed is determined by a matrix that is a multiple of the matrix hH.

Theorem 1. *In the class of spaces $\widetilde{\mathfrak{M}}$, the functional*

$$l_0^a(x) = 1 - \sum_{\gamma} h^n \delta(x - hH\gamma) \tag{9}$$

corresponds to a universally asymptotically optimal formula.

The proof is based on computing the norm of $l_0^a(x)$ and estimating the norm of $l_0(x)$ from below. This theorem was proved in a somewhat different way by T. Kh. Sharipov. Let us also note that this theorem is a generalization, to the case of arbitrary dimension, of a result of I. Babushka.

3°. We pass to the general case. Denote by $H^{(\mu)}$ the space of generalized functions $u(x)$ whose Fourier transforms $\hat{u}(\xi)$ are square-integrable with weight $\mu^2(\xi)$, and by $H^{(\mu)}(\Omega)$ the space of functions whose elements are restrictions of functions of the space $H^{(\mu)}$ to a bounded domain $\Omega \subset E_n$. The topology in these spaces is introduced as follows:

$$\|u\|_{H^{(\mu)}} = \left(\int_{E^n} |\hat{u}(\xi)|^2 \mu^2(\xi) \right)^{1/2} < \infty, \tag{10}$$

$$\|u\|_{H^{(\mu)}(\Omega)} = \inf \|u^c\|_{H^{(\mu)}}, \tag{11}$$

where the infimum is taken over all possible restrictions. Denote by $\mathfrak{M}_m$ the class of spaces $H^{(\mu)}$ whose weight functions $\mu(\xi)$ satisfy the conditions

$$\text{I.} \quad A(a + |\xi|^2)^{k/2} \leq \mu(\xi) \leq B(b + |\xi|^2)^{m/2}; \tag{12}$$

$A, B > 0$; $a, b \geq 0$; $k > n/2$; m is a positive integer. The constant m is fixed; the particular values of the remaining constants are immaterial.

$$\text{II.} \quad |D^{\alpha} F[\mu^{-2}(\xi)]| \leq K|x|^{2m-n-|\alpha|}; \tag{13}$$

F is the Fourier operator; α is the multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$;

$$|\alpha| = \sum_{i=1}^{n} \alpha_i, \qquad |x| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}.$$

We shall now make use of the cubature formulas with a regular boundary layer constructed by S. L. Sobolev [(2)]. Recall that, according to S. L. Sobolev, a formula is called a formula with a regular boundary layer of order p if its error functional admits the representation

$$l(x) = \sum l_{hH}^{(\gamma)}(x), \tag{14}$$

in which all component functionals have the form

$$l_{hH}^{(\gamma)}(x) = \mathcal{E}_{\Omega_{hH}^{(\gamma)}}(x) - \sum_{|\gamma'|<L} c_{\gamma'}^{(\gamma)} \delta(x - hH(\gamma + \gamma')) \tag{15}$$

($\Omega_{hH}^{(\gamma)}$ is the intersection of the domain obtained from the parallelepiped Ω_0, corresponding to the matrix H ($|H| = 1$), by shifting by the vector γ and dilating h-fold, with the domain Ω) and satisfy the conditions

$$\operatorname{supp}(l_{hH}^{(\gamma)}(x)) \subset \mathcal{E}(|x - hH\gamma| < Lh); \tag{16}$$

$$\|l_{hH}^{(\gamma)}(x)\|_{C^*} \leq Kh^n; \tag{17}$$

$$(l_{hH}^{(\gamma)}, x^{\alpha}) = 0, \qquad |\alpha| < p. \tag{18}$$

Let us note that in any such formula all coefficients corresponding to nodes located farther than $2Lh$ from the boundary coincide with one another and are equal to h^n. Therefore, in the sense of construction, these formulas are considerably simpler than optimal ones, since their coefficients need be computed only in the boundary layer.

Theorem 2. *Every formula with a regular boundary layer of order m is universally asymptotically optimal in the class of spaces $\mathfrak{M}_m$.*

The proof of this theorem is similar in idea to the proof of Theorem 1, but is considerably more complicated in detail. The necessary estimates of the norms of the functionals are based on the properties of the class $\mathfrak{M}_m$ and of formulas with a regular boundary layer.

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Received
11 VIII 1969

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.