SEVERAL NEW PROBABILITY INEQUALITIES CONNECTED WITH THE LÉVY METRIC

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Abstract

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SEVERAL NEW PROBABILITY INEQUALITIES CONNECTED WITH THE LÉVY METRIC

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The Lévy distance between two distribution functions F and G, defined by the equality

$$L(F,G)=\inf\{h:\ F(x-h)-h\le G(x)\le F(x+h)+h \text{ for all } x\},$$

as is known, is naturally connected with weak convergence of distributions. Recent works devoted to limit theorems for sums of independent random variables, in which the condition of limiting negligibility of the summands is not used (see, for example, ([1,2])), have demonstrated the exceptional role played in this theory by the Lévy metric.

At the same time, the use of the Lévy metric in probability theory has long been very limited for purely analytic reasons, primarily because of the lack of information about its connection with the apparatus of characteristic functions. Below we give several inequalities which, in our view, will help to increase specialists' interest in the Lévy metric.

Let f, g denote the characteristic functions corresponding to the distribution functions F and G.

Theorem 1. *For any distribution functions F, G and arbitrary numbers $T > e$, the inequality*

$$L(F,G)\le\frac{1}{\pi}\int_0^T|f(t)-g(t)|\,\frac{dt}{t}+2e\,\frac{\log T}{T}.$$

holds.

Let $\mathfrak{S}$ denote the set of all distribution functions concentrated in the interval $(-1/2,1/2)$, and let us agree henceforth to denote by θ the characteristic functions of distributions $\Theta\in\mathfrak{S}$.

Let $\psi(x)$ be a real-valued function defined on the positive half-line and possessing the properties:

1) $0 \le \psi(x) \le \psi(y)$ for $x \ge y$;
2) $\psi(xy) \le \psi(x)\psi(y)$.

Put

$$\Delta = \sup_{t>0} \psi(t)|f(t) - g(t)|, \tag{1}$$

$$C = \frac{1}{\pi} \int_0^\infty \frac{|\theta(t)|}{t\psi(t)}\, dt. \tag{2}$$

Theorem 2. *For any number $\varepsilon > 0$, any distribution functions F, G, and any function $\Theta \in \mathfrak{S}$, the inequality*

$$L(F, G) \le \varepsilon + C\Delta\psi(\varepsilon). \tag{3}$$

holds.

The proof of Theorems 1 and 2 was based on the use of the following inequalities.

Lemma. For any distribution functions F, G, H and arbitrary $\varepsilon > 0$, the inequalities

$$0 \le L(F, G) - L(F * H, G * H) \le 2\max(\varepsilon, H(-\varepsilon), 1 - H(\varepsilon)).$$

Corollary of Theorem 2. If, in addition, one requires the existence and monotonicity of the derivative of the function ψ, then the right-hand side of (3) is not hard to minimize with respect to ε. Namely, let $\varphi(x)$ denote the function inverse to $-\psi'(x)$. Then

$$L(F, G) \le \varphi\left(\frac{1}{C\Delta}\right) + C\Delta\psi\left[\varphi\left(\frac{1}{C\Delta}\right)\right]. \tag{4}$$

Let us note two special cases of inequality (4).

1°. If $\psi(x) = x^{-r}$, where $r > 0$, then

$$L(F, G) \le B\Delta^{1/(1+r)}; \qquad B = (1 + 1/r)(rC)^{1/(1+r)}, \tag{5}$$

and, as the elementary example shows,

$$F(x) = E(x); \qquad G(x) = (1 - \varepsilon)E(x) + \varepsilon E(x - \varepsilon), \quad 0 < \varepsilon < 1$$

(E is the unit distribution), the exponent $1/(1 + r)$ at Δ cannot be replaced by a larger constant.

As the function Θ it is convenient to take the N-fold convolution of the uniform distribution on $(-1/2N,\ 1/2N)$. If such functions are chosen, then for B it is not difficult to obtain the simple estimate

$$B < \frac{2}{r}(1+r)^2\pi^{-1/(1+r)}. \tag{6}$$

For small values of r, inequality (5) becomes rather ineffective, since, no matter how we choose the function $\Theta \in \mathfrak{S}$, whose characteristic function participates in the construction of C,

$$B \sim 1/\pi r, \quad \text{if } r \to 0.$$

Such behavior of B cannot be regarded as natural; most likely it is connected with the method. This shortcoming of the estimate can partly be made up for by choosing another family of functions ψ, for example the following one.

2°. Choose $\psi(x) = q^s(x)$, where $s > 1$ and

$$q(x) = \begin{cases} 2 - \log x, & \text{if } x < e, \\ 1/\log x, & \text{if } x \geq e. \end{cases}$$

This function belongs to the class of those considered by us, but it is not possible to use inequality (4) directly here, for the reason that for ψ there is no explicit form of the expression φ. However, if one assumes that Δ is sufficiently small, namely that

$$Cr\Delta \leq \exp(3 - e - s),$$

then one can write, already in explicit form, a somewhat cruder inequality than (4):

$$L(F,G) \leq C\Delta\psi(C\Delta).$$

As an illustration we present two inequalities obtained with the help of (5), which in our view are also of independent interest.

I. Let $K_F = \{F'\}$ be the set of all possible components of the distribution function F (including F itself and degenerate components). Fix a distribution G and define $\mathfrak{A}_\varepsilon$ as the set of those distributions F for which $L(F,G) \leq \varepsilon < 1$. Form the quantity

$$\beta_G(\varepsilon) = \sup_{\mathfrak{A}_\varepsilon} \sup_{K_F} \inf_{K_G} L(F', G').$$

It is known (see (^{3})) that for any distribution law G, $\beta_G(\varepsilon) \to 0$ if $\varepsilon \to 0$. However, constructing estimates of the quantity β_G is a very difficult problem. In the special case when $G = \Phi$ is the standard normal distribution, the author proved (^{3}) the existence of such positive numerical constants c_0, c_1 that *

$$c_0 \left(\log \frac{1}{\varepsilon}\right)^{-1/2} < \beta_\Phi(\varepsilon) < c_1 \left(\log \frac{1}{\varepsilon}\right)^{-1/11}.$$

Apparently, the correct estimate of the quantity β_Φ is

$$\beta_\Phi(\varepsilon) \asymp \left(\log \frac{1}{\varepsilon}\right)^{-1/2}.$$

Inequality (5) makes it possible, by a rather simple method, to improve the upper estimate of β_Φ.

Theorem 3. There exists a positive numerical constant c_2 such that

$$\beta_\Phi(\varepsilon) < c_2 \left(\log \frac{1}{\varepsilon}\right)^{-1/8}.$$

II. Let $F = F_1 * F_2 * \ldots$ and $G = G_1 * G_2 * \ldots$ be any decompositions of the distribution functions F and G into infinite compositions (the components may also be degenerate). Denote

$$\mu(k) = \sum_j \left| \int x^k \, d(F_j - G_j) \right|, \qquad \text{where } k \text{ is an integer } \geq 0,$$

$$\nu(r) = \sum_j \int |x|^r \, |d(F_j - G_j)|, \qquad \text{where } r \text{ is real } \geq 0,$$

$$\varkappa(r) = \sum_j \int |x|^{r-1} |F_j - G_j| \, dx, \qquad \text{where } r \text{ is real } \geq 1.$$

Theorem 4. If for some nonnegative integer m and some real $r \geq 1$, $m \leq r \leq m+1$, the conditions

1) $\mu(0) = \mu(1) = \ldots = \mu(m)$;

2) $\varkappa(r) < \infty$,

are satisfied, then the inequality

$$L(F, G) \leq D[\varkappa(r)]^{1/(1+r)} \tag{7}$$

holds, where

$$D = (1 + 1/r)(rCW)^{1/(1+r)}$$

and

$$W = 1, \quad \text{if } m = 0,$$

$$W = (2m)^{m+1-r}/\Gamma(m+1), \quad \text{if } m \geq 1.$$

By a special choice of the function θ, one can obtain an absolute estimate for D. Namely,

$$D \leq 8.1.$$

The same example that was mentioned in connection with inequality (5) shows that there exist such distributions F and G for which

$$L(F, G) > \exp(-1/e)[\varkappa(r)]^{1/(1+r)}.$$

We note that inequality (7) makes it possible to obtain an analogous inequality using the quantity $\nu(r)$, since for $r \geq 1$

$$r\varkappa(r) \leq \nu(r). \tag{8}$$

The inequalities between L and the quantities $\varkappa, \nu$, as is easy to see, realize relations between the Lévy metric and metrics of another type—the mean metric with weight and the variation with weight.

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CITED LITERATURE

[1] Yu. V. Linnik, *Decompositions of Probability Laws*, 1960. [2] V. M. Zolotarev, *Theory of Probability and Its Applications*, **12**, 4 (1967). [3] V. M. Zolotarev, ibid., **13**, 4 (1968).

[*] A. G. Maloshevsky, in his Candidate dissertation, was able to improve the upper estimate somewhat by replacing 1/11 by 1/10.

Note: Figure translations are in progress. See original paper for figures.

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