ON LAGRANGIAN INVARIANTS OF THE DYNAMICS OF AN INVISCID COMPRESSIBLE FLUID

HYDROMECHANICS

1970

SovietRxiv

Abstract

Full Text

UDC 532.511

HYDROMECHANICS

B. L. GAVRILIN, M. M. ZASLAVSKII

ON LAGRANGIAN INVARIANTS OF THE DYNAMICS OF AN INVISCID COMPRESSIBLE FLUID

(Presented by Academician L. I. Sedov on 29 IX 1969)

1. In various problems of hydrodynamics, invariant hydrodynamic characteristics specific to a continuous medium are used, the values of which, for a fixed fluid particle, remain unchanged during its motion. The Eulerian fields of such invariant hydrodynamic characteristics $I(x,t)$ (called below Lagrangian invariants) satisfy a differential equation of the form (1)

$$\frac{dI(x,t)}{dt}=\frac{\partial I}{\partial t}+\frac{\partial I}{\partial x^i}v^i=0 \quad (i=1,2,3), \tag{1}$$

a solution of which may serve as an independent definition of a Lagrangian invariant in Eulerian coordinates. Obvious examples of Lagrangian invariants are the Eulerian fields of the initial coordinates of particles of the medium $x_0^i(x,t)$, the initial velocities $v_0^i(x,t)$, and also the entropy $s(x,t)$ in adiabatic reversible processes in a continuous medium. For the adiabatic model of an inviscid compressible fluid, Ertel (2) obtained the Lagrangian invariant $\bar{I}_1^a(x,t)=\alpha[\nabla s(\nabla\times v)]$ $(\alpha=1/\rho,\ \rho=\rho(x,t)$ is the density). For the same model, Hollmann (3) found three more Lagrangian invariants

$$\bar{I}_2^a(x,t)=\alpha[(v-\nabla\psi^a)(\nabla s\times\nabla\bar{I}_1^a)],$$

$$\bar{I}_3^a(x,t)=\alpha[(v-\nabla\psi^a)(\nabla s\times\nabla\bar{I}_2^a)],$$

$$\bar{I}_4^a(x,t)=\alpha[\nabla\bar{I}_2^a(\nabla\bar{I}_1^a\times\nabla s)]$$

$$\left(\psi^a=\int_0^t\left(\frac{1}{2}v^2-u-h\right)dt,\right.$$

$u = u(x,t)$ is the specific potential of external body forces, $h = h(s,P)$ is the specific enthalpy, $P = P(x,t)$ is the pressure). For the barotropic model, Ertel and Rossby (4) obtained a Lagrangian invariant of the form

$$\bar{I}_1^b(x,t) = \alpha[(v - \nabla\psi^b)(\nabla \times v)], \qquad \text{where } \psi^b = \int_0^t \left(\frac{1}{2}v^2 - u - \int \frac{dP}{\rho}\right) dt.$$

The question naturally arises of systematizing the known Lagrangian invariants, the greater part of which is usually derived from the equations of motion by means of rather artificial devices. Connected with this question is also the more general problem of developing a general method for constructing a complete set of Lagrangian invariants for the corresponding closed system of hydrodynamic equations. Below, this range of questions is considered for three models of an inviscid compressible fluid—barotropic, adiabatic, and isothermal. Since any function of Lagrangian invariants $I_1, I_2, \dots, I_n$ is also a Lagrangian invariant, it is therefore natural to be interested only in sets of Lagrangian invariants, defined up to arbitrary transformations of the form $I_1, I_2, \dots, I_n \to f(I_1, I_2, \dots, I_n)$.

2. Let us consider the Lagrangian invariants $x_0^i(x,t)$, $s(x,t)$, and show that these four Lagrangian invariants, for a known explicit expression of some thermodynamic potential—for example, the specific internal energy $e(\rho, s)$ —completely determine the sought Eulerian fields of variables $v^i(x,t)$, $P(x,t)$, $\rho(x,t)$, which, together with $s(x,t)$, enter the system of Eulerian equations of the adiabatic model of an inviscid compressible fluid (the three equations of momentum balance, the continuity equation, the adiabaticity equation, and the equation of state determined by the prescribed expression for $e(\rho,s)$). To prove this fact we use the generalized Weber transformation (^{5}), by means of which one can obtain from the system of equations for the adiabatic model the following representation of the velocity field in terms of the Lagrangian invariants $x_0^i(x,t)$, $s(x,t)$:

$$v^i(x,t) = \partial\psi^a/\partial x^i + \beta^a \partial s/\partial x^i + v_0^j \partial x_0^i/\partial x^j, \tag{2}$$

where $v_0^j = \partial x_0^j/\partial t$, $\beta^a = \int_0^t T\,dt$, $T = \partial e/\partial s$ is the temperature, $\psi^a = \int_0^t (1/2v^2 - u -$

$$-e - p/\rho)\,dt.$$

Adding to (2) the continuity equation for a prescribed potential of external forces u, we obtain a closed system of equations for determining the sought Eulerian fields $v^i(x,t)$, $p(x,t)$, $\rho(x,t)$ from the prescribed Lagrangian invariants $x_0^i(x,t)$ and $s(x,t)$. This proves the validity of the assertion made.

It should be noted that the introduction of the Lagrangian invariants $x_0^i(x,t)$ when finding a representation of the velocity field in the form (2) is, to a certain extent, an arbitrary device. What is essential is only the fact that the Eulerian velocity field in the adiabatic model of an inviscid compressible fluid is expressed in terms of three Lagrangian invariants determined by the initial conditions, and also in terms of the Lagrangian invariant $s(x,t)$, which specifies the adiabatic model. One can, for example, obtain a representation of the velocity field in a form analogous to (2), starting from the triple of Lagrangian invariants $v_0^i(x,t)$ and $s(x,t)$. In particular, it proves convenient to introduce the triple of Lagrangian invariants $\varphi_0(x,t)$, $\lambda_0(x,t)$, $\mu_0(x,t)$, which appear when the initial velocity field is represented in the form $v_0^i(x,t) = v_0^i(x_0) = \partial\varphi_0/\partial x_0^i + \lambda_0 \partial\mu_0/\partial x_0^i$. In this case the Eulerian velocity field is expressed through $\varphi_0, \lambda_0, \mu_0, s$ in the following way:

$$v^i(x,t) = \frac{\partial \varphi^a}{\partial x^i} + \beta^a \frac{\partial s}{\partial x^i} + \lambda_0 \frac{\partial \mu_0}{\partial x^i}, \qquad \varphi^a = \varphi_0 - \psi^a. \tag{3}$$

(This relation is a generalization of the Clebsch transformation (^{5}) of the velocity field known for the barotropic model.)

It is not difficult to show further (using only the continuity equation and arguments connected with the transformation properties of Lagrangian coordinates) that if some functions $I_1(x,t)$, $I_2(x,t)$, $I_3(x,t)$ are Lagrangian invariants, then the expression

$$I(x,t) = \alpha \frac{\partial(I_1, I_2, I_3)}{\partial(x^1, x^2, x^3)} = \alpha \left[\nabla I_1 (\nabla I_2 \times \nabla I_3)\right]. \tag{4}$$

is also a Lagrangian invariant.

This relation may be taken as the basis of a general method for constructing a system of Lagrangian invariants for the adiabatic model from four given initial Lagrangian invariants.

3. The form of expression (4) suggests that the most suitable triple of Lagrangian invariants determined by the initial conditions,

are the Lagrangian invariants appearing in the generalized Clebsch representation (3). In constructing a set of Lagrangian invariants for the adiabatic model, it is expedient to define the notion of the orders of Lagrangian invariants. Namely, by Lagrangian invariants of the first, second, etc., orders we shall mean Lagrangian invariants containing, respectively, first, second, etc., derivatives with respect to the spatial coordinates of the initial Lagrangian invariants. We shall also assume that these derivatives exist.

With the aid of relation (4) one can construct a series of $C_4^3 = 4$ Lagrangian invariants of first order from the four initial Lagrangian invariants $I_0^a = s,\ I_1^a = \varphi_0,\ I_2^a = \lambda_0,\ I_3^a = \mu_0$:

$$I_4^a = \alpha[\nabla s(\nabla\lambda_0 \times \nabla\mu_0)], \qquad I_5^a = \alpha[\nabla\varphi_0(\nabla\lambda_0 \times \nabla\mu_0)],$$

$$I_6^a = \alpha[\nabla\varphi_0(\nabla s \times \nabla\mu_0)], \qquad I_7^a = \alpha[\nabla s(\nabla\varphi_0 \times \nabla\lambda_0)].$$

The first of these, as is not hard to verify, can be reduced to the form of Ertel' s Lagrangian invariant $\bar{I}_1^a$. The remaining Lagrangian invariants of first order for the adiabatic model of an inviscid compressible fluid, apparently, have not previously been considered. In particular, the Lagrangian invariant I_5^a can be reduced to the following form, more convenient for applications:

$$I_5^a = \alpha[(v - \nabla\varphi^2) - \beta^a\nabla s][(\nabla \times v) - (\nabla\beta^a \times \nabla s)].$$

Using the same procedure, one can further obtain, from the 4 initial Lagrangian invariants and 4 Lagrangian invariants of first order, the following series of $C_4^3 + 2C_4^1C_4^2 = 52$ Lagrangian invariants of second order. This series includes precisely the first of the three invariants $\bar{I}_3^a$ obtained by Hollmann. The number of Lagrangian invariants of third order for the adiabatic model is equal to $C_{52}^3 + C_8^2C_{52}^1 + C_8^1C_{52}^2 = 34608$ and is practically impossible to survey. They include the two other Hollmann invariants $\bar{I}_3^a$, $\bar{I}_4^a$. We shall not dwell on the expressions of $\bar{I}_2^a$, $\bar{I}_3^a$, $\bar{I}_4^a$ in terms of the Lagrangian invariants of the general formalism presented here, because of their bulkiness. Lagrangian invariants of higher orders are of even less interest, although their formal writing presents no difficulty.

Let us also note the changes in the system of Lagrangian invariants for the adiabatic model that arise when passing to the two-dimensional case. Instead of (3) and (4) we shall then have the relations

$$v^\alpha(x^1, x^2, t) = \frac{\partial\psi^a}{\partial x^\alpha} + \beta^a\frac{\partial s}{\partial x^\alpha} + \lambda_0\frac{\partial\mu_0}{\partial x^\alpha}, \qquad \alpha = 1, 2;$$

$$I(x^1, x^2, t) = \alpha[\nabla_2 I_1 \times \nabla_2 I_2], \qquad \nabla_2 = \frac{\partial}{\partial x^1}i + \frac{\partial}{\partial x^2}j.$$

Consequently, in the two-dimensional adiabatic model there exist only three Lagrangian invariants of first order ($C_3^2 = 3$). Choosing as the initial Lagrangian invariants $I_0^a = s(x^1, x^2, t)$, $I_1^a = \lambda_0(x^1, x^2, t)$, $I_2^a = \mu_0(x^1, x^2, t)$, we obtain the following series of Lagrangian invariants of first order:

$$I_3^a = \alpha[\nabla_2 s \times \nabla_2\lambda_0], \qquad I_4^a = \alpha[v \times \nabla_2 s],$$

$$I_5^a = \alpha[\nabla_2 \times v - \nabla_2\beta^a \times \nabla_2 s]$$

(the last of these is a generalization of the usual vorticity conservation law in the two-dimensional barotropic model). We shall not dwell here on Lagrangian invariants of higher orders.

In order, in terms of Lagrangian invariants, to distinguish the cases of the barotropic and adiabatic models, we introduce the following definition: a Lagrangian invariant $I(x,t)$ will be called trivial if, in addition to the general condition (1), $\partial I/\partial x^i = 0$, so that $I = \text{const}$. In connection with this definition, we note that when considering Lagrangian invariants for the adi-

for the adiabatic model, all the initial Lagrangian invariants determined by the initial conditions were assumed to be nontrivial. (We shall not dwell here on the obvious changes in the system of invariants for the adiabatic model that are associated with abandoning this assumption.)

It is clear that for the barotropic model of an inviscid compressible fluid there corresponds the specification of the initial trivial Lagrangian invariant $s(x,t) = \text{const}$. Owing to this circumstance, among the first-order Lagrangian invariants in the barotropic case only one Lagrangian invariant proves to be nontrivial, coinciding, as is easily verified, with the Ertel—Rossby invariant $\bar{I}_1^b(x,t)$. The number of nontrivial second-order Lagrangian invariants, in connection with the same circumstance, decreases from 52 to $C_4^3 = 3$, etc. For the two-dimensional barotropic model, analogous considerations lead to the conclusion that there exists a single nontrivial invariant of first order (coinciding with the ordinary vorticity conservation law), two nontrivial Lagrangian invariants of second order, etc.

In conclusion we note that Lagrangian invariants for the isothermal model of an inviscid compressible fluid can be obtained from the relations considered above for the adiabatic model by the formal replacement of the initial nontrivial Lagrangian invariant $I_0^a = s(x,t)$ by $I_0^c = T(x,t)$, while in (2) β^a is replaced by

$$\beta^c = -\int_0^t s\,dt,$$

ψ^a by

$$\psi^c = \int_0^t ({}^1/{}_2 v^2 - z - u)\,dt$$

($z = z(T,P)$ is the specific thermodynamic Gibbs potential). With these changes taken into account, all the results obtained for the adiabatic model remain valid.

The authors express their deep gratitude to A. S. Monin for his attention to the work and for valuable comments.

Institute of Oceanology named after P. P. Shirshov
Academy of Sciences of the USSR
Moscow

Received
24 VI 1969

CITED LITERATURE

[1] A. S. Monin, A. M. Yaglom, *Statistical Hydromechanics*, Part I, Moscow, 1965.
[2] H. Ertel, Meteor. Zs., **59**, 277 (1942).
[3] G. Hollmann, Arch. Meteorol., Geophys. u. Bioklimatol., A, **14**, No. 1, 1 (1964).
[4] H. Ertel, G. Rossby, Sitzungs-Ber. Dtsch. Akad. Wiss. Berlin, Kl. f. Math. u. allg. Naturwiss., No. 1 (1949).
[5] J. Serrin, *Mathematical Foundations of Classical Fluid Mechanics*, Moscow, 1963.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.