ON NILPOTENT AND SUPERSOLVABLE SUBGROUPS OF FINITE GROUPS

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Abstract

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§ 1. A finite group all of whose Sylow subgroups are invariant is called, as is known, special or nilpotent. In 1929, in the paper ([1]), we obtained the following theorems:

I. *If for every primary divisor* (definition below) $\delta > 1$ *of the order of the commutator subgroup of a finite group of order* g *the condition* $(\delta - 1, g) = 1$ *is satisfied, then the group is special.*

II. *If for every primary divisor* $\delta > 1$ *of a natural number* g *the condition* $(\delta - 1, g) = 1$ *is satisfied, then all groups of order* g *are special.*

Thirty years later Theorem II was obtained again by G. Pazderski ([2]), to whom our paper ([1]), apparently, remained unknown. The paper ([1]) was continued by us in ([3], [4]). The basic arithmetical idea of ([1]) and its continuations ([3], [4]), in combination with the main feature of our method of indices ([5]) and factorizations of groups ([9]) (to look for subgroups whose orders are products of the indices of certain series of subgroups), we now apply to the detection of nilpotent and supersolvable subgroups of a finite group. Let us note the simplicity of formulation of Theorem 2 given below.

§ 2. **Notation.** A primary divisor of a natural number is a divisor that is a power of a prime number (including 1); if F is a finite sequence of natural numbers, then $\overline{F}$, when F is nonempty, denotes the product of all elements of F, and when F is empty is equal to 1; $\mathfrak{G}$ is a finite group; $|\mathfrak{G}|$ is its order; k is the order of its commutator subgroup; $\mathfrak{E}$ is its identity subgroup; an invariant series of $\mathfrak{G}$ is a series of subgroups all of whose terms are invariant in $\mathfrak{G}$; a refinement of an invariant series R of the group $\mathfrak{G}$ (see ([5]))

$$R_f : \quad \mathfrak{G} = \mathfrak{G}_0 \supseteq \mathfrak{G}_1 \supseteq \cdots \supseteq \mathfrak{G}_{\beta-1} \supseteq \mathfrak{F}_\beta \supseteq \mathfrak{G}_\beta \supseteq \mathfrak{F}_{\beta+1} \supseteq \mathfrak{G}_{\beta+1} \supseteq \cdots \cdots \mathfrak{G}_{\nu-1} \supseteq \mathfrak{F}_\nu \supseteq \mathfrak{G} = \mathfrak{E}$$

by means of subgroups $\mathfrak{F}_i$, $i = \beta, \dots, \nu$, we shall call an index series of $\mathfrak{G}$ if each $\mathfrak{F}_i$ satisfies the "conjugacy condition" : every subgroup of $\mathfrak{G}_{i-1}$ conjugate to $\mathfrak{F}_i$ in $\mathfrak{F}_\beta$ is already conjugate to $\mathfrak{F}_i$ in $\mathfrak{G}_{i-1}$; the subgroups $\mathfrak{F}_i$ will be called the factorial subgroups of the series R or of the series R_f; if $|\mathfrak{F}_i/\mathfrak{G}_i| = f_i$, then

$f_\beta f_{\beta+1} \cdots f_\nu = h$ will be called the index of the group $\mathfrak{G}$ or the index of the series R, and $\mathfrak{F}_i/\mathfrak{G}_i$ its factors; if $\mathfrak{A}/\mathfrak{B}$, then $\mathfrak{A}$ will be called the numerator and $\mathfrak{B}$ the denominator of the factor group $\mathfrak{A}/\mathfrak{B}$; a subgroup $\tilde{\mathfrak{F}}_i$, $\beta \leq n \leq \nu$, of $\mathfrak{F}_\beta$ will be called a nilpotent d-extension (a supersolvable d-extension) of $\mathfrak{F}_i$ in $\mathfrak{F}_\beta$, if $\mathfrak{F}_i$ is invariant in $\tilde{\mathfrak{F}}_i$ and $\tilde{\mathfrak{F}}_i/\mathfrak{F}_i$ is a nilpotent group (a supersolvable group) of order d; Schmidt groups are minimal non-nilpotent groups ([10]); a maximal subgroup is, when $\mathfrak{G} \neq \mathfrak{E}$, a proper subgroup of $\mathfrak{G}$ (i.e. $\neq \mathfrak{G}$) that is not a proper subgroup of any proper subgroup of $\mathfrak{G}$ (and when $\mathfrak{G} = \mathfrak{E}$, it is $\mathfrak{E}$ itself); $E(\mathfrak{G}, n, \mathfrak{A}, a)$ means that $\mathfrak{G}$ has a nilpotent subgroup $\mathfrak{A}$ and $|\mathfrak{A}| = a$; $E(\mathfrak{G}, n, a)$ means that $\mathfrak{G}$ has a nilpotent subgroup of order a.

§ 3. **Definition 1.** Let r and $\delta_1, \delta_2, \dots, \delta_m$ be natural numbers and let $H_i = \{\delta_1, \delta_2, \dots, \delta_i\}$, $1 \leq i \leq m$. Then $H = H_m$ will be called a sequence of type rC, or an rC-sequence, if for every $i = 1, \dots, m$ and every primary divisor d_i of the number (δ_i, r) it follows from $d_i > 1$ that $(d_i - 1, \overline{H}_i) = 1$. For $m = 1$, an rC-sequence will be called an rC-number. In the notation rC-sequence and rC-number we shall agree to omit r when $r = \delta_1, \delta_2 \dots \delta_m$. The empty sequence will also be assigned to type rC.

Theorem 1. Let d be any natural number. If the kdC-sequence (the C-sequence) H is a subsequence of the sequence of all indices of some invariant series R of the group $\mathfrak{G}$, then $\mathfrak{G}$ has a nilpotent subgroup $\mathfrak{H}$ of order $\overline{H}$. If $(\overline{H}, k) = 1$, or if $\overline{H}$ is free of cubes of primes, then $\mathfrak{H}$ is abelian. If R is a principal series of $\mathfrak{G}$, then all terms H are primary numbers.

Proof. It is obvious that it suffices to prove the theorem for the case when H is a kC-sequence and the series R is principal. By hypothesis, each term h of H is a kC-number. Hence, taking into account that the order of the commutant of the corresponding factor group of the series R divides k, we conclude, by our theorem I, that this factor group will be a nilpotent group. But, as a factor of a principal series, it is elementary. Therefore its order h is a primary number. Now let $\mathfrak{G}$ be a counterexample of least order to the theorem being proved, and let H be that kC-sequence for which $\mathfrak{G}$ has no corresponding subgroup. Then $\overline{H} > 1$, i.e. $|\mathfrak{G}| > 1$. Let

$$\mathfrak{G} = \mathfrak{G}_0 \supset \mathfrak{G}_1 \supset \cdots \supset \mathfrak{G}_\lambda = \mathfrak{E}$$

be the series R, and let $|\mathfrak{G}_{\lambda-1}| = h_\lambda$. Of course, $h_\lambda > 1$.

1) $h_\lambda \in H$. The theorem is obviously true for $\mathfrak{G}/\mathfrak{G}_{\lambda-1}$ with respect to the kC-sequence $H \setminus \{h_\lambda\}$. Therefore $\mathfrak{G}/\mathfrak{G}_{\lambda-1}$ contains a nilpotent subgroup $\mathfrak{N}/\mathfrak{G}_{\lambda-1}$ of order $\overline{H}/h_\lambda$. By the preceding, $h_\lambda = q^\omega$, $\omega > 0$, q a prime. If $\overline{H}/h_\lambda = 1$, then $H = \{h_\lambda\}$, and $\mathfrak{G}_{\lambda-1}$ will be the desired nilpotent subgroup. Let $\overline{H}/h_\lambda > 1$, and let p be an arbitrary prime divisor of the number $\overline{H}/h_\lambda$. Since $\mathfrak{N}/\mathfrak{G}_{\lambda-1}$ is a nilpotent group, its Sylow p-subgroup $\mathfrak{P}/\mathfrak{G}_{\lambda-1}$ of order p^α, $\alpha > 0$, is invariant in it. Then $\mathfrak{P}$ of order $p^\alpha q^\omega$ is invariant in $\mathfrak{N}$. Suppose $p \neq q$. If a Sylow p-subgroup from $\mathfrak{P}$ is invariant in it, then it will be an invariant Sylow p-subgroup of $\mathfrak{N}$. Suppose a Sylow

p-subgroup from $\mathfrak{P}$ is not invariant in it. Then, by theorem 3.2 from (1), it follows that $\mathfrak{P}$ contains a Schmidt p-nilpotent group $\mathfrak{S}$ of order $p^{\alpha_1}q^{\omega_1}$, $\alpha_1, \omega_1 > 0$. By lemma 1 from (1), q^{ω_1} is the order of the commutant of $\mathfrak{S}$, and therefore q^{ω_1} divides k. By hypothesis, $h_\lambda = q^\omega \in H$, and p divides $\overline{H}/h_\lambda$. Hence, taking into account that H is a kC-sequence, it follows that $q^x \not\equiv 1 \pmod p$ for all $1 \leq x \leq \varphi_1$. But then the Sylow p-subgroup from $\mathfrak{S}$ is invariant in it, which is impossible. Suppose $p = q$. Then $\mathfrak{P}$ will be an invariant q-Sylow subgroup in $\mathfrak{N}$. Thus, $E(\mathfrak{G}, n, \mathfrak{N}, \overline{H})$. A contradiction.

2) $h \in H$. The theorem is obviously true for $\mathfrak{G}/\mathfrak{G}_{\lambda-1}$ with respect to the kC-sequence H. Therefore $\mathfrak{G}/\mathfrak{G}_{\lambda-1}$ contains a nilpotent subgroup $\mathfrak{N}/\mathfrak{G}_{\lambda-1}$ of order $\overline{H} > 1$. For $\mathfrak{N}/\mathfrak{G}_{\lambda-1}$, one can obviously construct a principal series with a nondecreasing sequence of prime indices F, and moreover $\overline{F} = \overline{H}$.

2.1) There exists a maximal subgroup $\mathfrak{M}$ of the group $\mathfrak{N}$ not containing $\mathfrak{G}_{\lambda-1}$. Then $\mathfrak{N} = \mathfrak{M}\mathfrak{G}_{\lambda-1}$. In view of $\mathfrak{N}/\mathfrak{G}_{\lambda-1} \simeq \mathfrak{M}/\mathfrak{M} \cap \mathfrak{G}_{\lambda-1}$ and $|\mathfrak{M}| < |\mathfrak{G}|$, the theorem is true for $\mathfrak{M}$ with respect to the above-mentioned kC-sequence F. Therefore $E(\mathfrak{M}, n, F)$, $\overline{F} = \overline{H}$. A contradiction.

2.2) $\mathfrak{G}_{\lambda-1}$ is contained in every maximal subgroup $\mathfrak{M}$ of the group $\mathfrak{N}$. Then $\mathfrak{M}/\mathfrak{G}_{\lambda-1}$ will be a maximal subgroup of the group $\mathfrak{N}/\mathfrak{G}_{\lambda-1}$. But $\mathfrak{N}/\mathfrak{G}_{\lambda-1}$ is a nilpotent group, and therefore $\mathfrak{M}/\mathfrak{G}_{\lambda-1}$ is invariant in it. This means that all maximal subgroups $\mathfrak{M}$ of $\mathfrak{N}$ are invariant in it. Then ([6]) the group $\mathfrak{N}$ of order $\overline{H}h_\lambda$ is a nilpotent group, and therefore $E(\mathfrak{N}, n, \overline{H})$. A contradiction. The assertions about the abelianness of the subgroup of order $\overline{H}$ are obvious.

Theorem 2. If a nondecreasing sequence of prime numbers P is a subsequence of the sequence of all indices of some principal series of the group $\mathfrak{G}$, then $\mathfrak{G}$ has a nilpotent subgroup of order $\overline{P}$.

Proof. Apply theorem 1 to P.

§ 4. Let

$$R_i: \ \mathfrak{F}_i = \mathfrak{F}_i^{(0)} \supset \mathfrak{F}_i^{(1)} \supset \cdots \supset \mathfrak{F}_i^{(\nu_i)} = \mathfrak{G}_i, \quad i = \beta, \dots, \nu$$

be a series of invariant subgroups of the factorial subgroup $\mathfrak{F}_i$, and let Φ_i be some subsequence of the sequence of all factors of the invariant

of a series of subgroups R_i (Φ_i may, in particular, be empty). Let a_i also be the sequence of the orders of all elements of Φ_i. The numerators and denominators of all factor groups from Φ_i form a subsequence A_i of the sequence R_i. We shall call the sequence of subgroups A_i an insertion of type a_i of the group $\mathfrak{F}_i$. The numerator and denominator of each term of Φ_i will be called a diad from the insertion A_i. Let

$$A_i^* = A_\beta \cup A_{\beta+1} \cup \cdots \cup A_i$$

and

$$a_i^* = a_\beta \cup a_{\beta+1} \cup \cdots \cup a_i.$$

Then A_i^* will be called an insertion of type a_i^* (of the series R_f or of the group $\mathfrak{G}$). We shall regard a_i^* as given if, for $j = \beta, \dots, i$, the insertion A_i is specified together with its division into diads. For $i = \nu$ put $A^*_\nu = A$ and $a^*_\nu = a$. A diad from any A_i will also be called a diad from A. For $i < \nu$ one may regard A^*_i as such an A for which $a_{i+1}, a_{i+2}, \dots, a_\nu$ are empty.

Definition 2. The insertion A^*_i, $i = \beta, \dots, \nu$, will be called an r-insertion (a supersoluble insertion) if: 1) a^*_i is an rC-sequence (a sequence of prime numbers); 2) if $i > \beta$, then for $j = \beta + 1, \dots, i$ every subgroup from A_j is invariant in any nilpotent $\overline{a}^*_{j-1}$-extension (supersoluble $\overline{a}^*_{j-1}$-extension) of the subgroup $\mathfrak{F}_j$ in $\mathfrak{F}_\beta$.

Definition 3. The type a^*_i of a kd-insertion (supersoluble insertion) A^*_i, $i = \beta, \dots, \nu$, will be called a nilpotent (supersoluble) sequence of index h of the group $\mathfrak{G}$ (d natural).

Of interest, because of its simplicity, is the case $kd = |\mathfrak{G}|$, which does not require knowledge of the number k.

An example of a nilpotent sequence a is obtained if, for $i = \beta, \dots, \nu$, all subgroups from A_i are characteristic in $\mathfrak{F}_i$, and a is a nondecreasing sequence of primes. Then a is a nilpotent sequence. Thus, if all factors of an indexial h are cyclic and, for every $i > \beta$, the greatest prime divisor of the number f_i is not greater than the least prime divisor of the number f_{i-1}, then $h = \overline{a}$. Such an indexial will be called cyclically nondecreasing.

§ 5. Theorem 3. *If a is a nilpotent supersoluble sequence of indexial h of the group $\mathfrak{G}$, then $\mathfrak{G}$ has a nilpotent supersoluble subgroup $\mathfrak{H} \leq \mathfrak{F}_\beta$ of order $\overline{a}$.*

Proof. We first prove the theorem for the case of nilpotent sequences. Let $\mathfrak{G}$ be a counterexample of least order to the theorem being proved. Then $\mathfrak{G}$ has no nilpotent subgroup of order $\overline{a}$, where a is the type of some kd-insertion A of some series of the form R_f. This means that $\overline{a} > 1$, whence also $|\mathfrak{G}| > 1$. Then in the series R there exists $\mathfrak{G}_\lambda = \mathfrak{G}$ with the least number λ. Therefore $\mathfrak{G}_{\lambda-1} \neq \mathfrak{G}$ and $\overline{a}_{\lambda+1} = \overline{a}_{\lambda+2} = \dots = \overline{a}_\nu = 1$. For $\beta = \lambda$ we have $\overline{a} = \overline{a}_\beta$, where a_β is a kdC-sequence of certain indices of the invariant series A_λ of the group $\mathfrak{F}_\lambda$. By Theorem 1, $E(\mathfrak{F}_\lambda, n, \overline{a})$. A contradiction. Hence $\beta < \lambda - 1$. Let

$$L: \quad \mathfrak{F}_\beta \supseteq \dots \supseteq \mathfrak{G}_{\lambda-1}$$

be a section of the series R_f, refined by the r-insertion $A^*_{\lambda-1}$ of type $a^*_{\lambda-1}$ (r arbitrary). Factoring all terms of L by $\mathfrak{G}_{\lambda-1}$, we obtain the series

$$M: \quad \mathfrak{F}_\beta/\mathfrak{G}_{\lambda-1} \supseteq \dots \supseteq \mathfrak{G}_{\lambda-1}/\mathfrak{G}_{\lambda-1}.$$

A direct verification shows that (a) is true: if the series L has an r-insertion of type $a^*_{\lambda-1}$, then M also has an r-insertion of the same type.

a) $|\mathfrak{F}_\lambda| = 1$. Taking account of (a) for $r = kd$ and of the fact that the order of the commutator subgroup of $\mathfrak{F}_\beta/\mathfrak{G}_{\lambda-1}$ divides k, and

$$|\mathfrak{F}_\beta/\mathfrak{G}_{\lambda-1}| < |\mathfrak{G}|,$$

we see that the theorem for $\mathfrak{F}_\beta/\mathfrak{G}_{\lambda-1}$ is valid with respect to $a^*_{\lambda-1}$. Therefore

$$E(\mathfrak{F}_\beta/\mathfrak{G}_{\lambda-1}, n, \mathfrak{R}/\mathfrak{G}_{\lambda-1}, \overline{a}^*_{\lambda-1}).$$

But, since $\mathfrak{F}_\lambda = \mathfrak{G}$, $\overline{a}_\lambda = 1$. Hence $\overline{a}^*_{\lambda-1} = \overline{a}$, i.e.

$$|\mathfrak{R}/\mathfrak{G}_{\lambda-1}| = \overline{a}.$$

Then $\mathfrak{R}/\mathfrak{G}_{\lambda-1}$ has a chief series with a nondecreasing sequence of prime indices, i.e. $\mathfrak{R}$ has an invariant series passing through $\mathfrak{G}_{\lambda-1}$ and having, on the section from $\mathfrak{R}$ to $\mathfrak{G}_{\lambda-1}$, a nondecreasing sequence of prime indices whose product is equal to $\overline{a}$. Hence, by Theorem 2, we conclude that $E(\mathfrak{R}, n, \overline{a})$. A contradiction.

b) $|\mathfrak{F}_\lambda| > 1$ and $\mathfrak{F}_\lambda$ is invariant in $\mathfrak{F}_\beta$. As in case a), we ascertain the existence of

$$E(\mathfrak{F}_\beta/\mathfrak{G}_{\lambda-1}, n, \mathfrak{R}/\mathfrak{G}_{\lambda-1}, \overline{a}^*_{\lambda-1})$$

and of an invariant series

$$\mathfrak{R} \supseteq \cdots \supseteq \mathfrak{G}_{\lambda-1} \supseteq \mathfrak{F}_\lambda \supseteq \mathfrak{G},$$

in which, from $\mathfrak{R}$ to $\mathfrak{G}_{\lambda-1}$, all indices form a nondecreasing

sequence σ of prime numbers, with $\bar{\sigma} = \bar{a}^*_{\lambda-1}$. But then σ will also be the sequence of indices of the invariant series

$$\mathfrak{N}/\mathfrak{F}_\lambda \supseteq \cdots \supseteq \mathfrak{G}_{\lambda-1}/\mathfrak{F}_\lambda \supseteq \mathfrak{F}_\lambda/\mathfrak{F}_\lambda$$

on the segment from $\mathfrak{N}/\mathfrak{F}_\lambda$ to $\mathfrak{G}_{\lambda-1}/\mathfrak{F}_\lambda$. Hence, by Theorem 2, we conclude that

$$E(\mathfrak{N}/\mathfrak{F}_\lambda, n, \mathfrak{N}^*/\mathfrak{F}_\lambda, \bar{a}^*_{\lambda-1}).$$

Then there exists an invariant series

$$S: \quad \mathfrak{N}^* \supseteq \cdots \supseteq \mathfrak{F}_\lambda \supseteq \mathfrak{G}$$

with sequence of indices σ on the segment from $\mathfrak{N}^*$ to $\mathfrak{F}_\lambda$. We refine the series S on the segment from $\mathfrak{F}_\lambda$ to $\mathfrak{G}$ by the kd-insertion A_λ and obtain the series S^*. All subgroups from A_λ, according to 2) of Definition 2, will be invariant in $\mathfrak{N}^*$. Further, the sequence of those indices of the series A_λ which enter into a_λ will consist, according to Definition 1 and 1) of Definition 2, of the numbers δ_j with the following property: every prime divisor > 1, d_j, of the number (δ_j, kd) is such that

$$(d_j - 1, \bar{a}^*_{\lambda-1}\bar{a}_{\lambda j}) = 1,$$

where $a_{\lambda j}$ denotes the sequence of all members of a_λ which precede δ_j, and δ_j itself. But

$$\bar{a}^*_{\lambda-1} = \bar{\sigma}.$$

Then

$$(d_j - 1, \bar{\sigma}\bar{a}_{\lambda j}) = 1.$$

This shows that the sequence $\bar{H}$, composed of all indices of the series S^* on the segment from $\mathfrak{N}^*$ to $\mathfrak{F}_\lambda$ and of all indices of the series A_λ entering into a_λ, will be a kdC-sequence, and moreover

$$\bar{H} = \bar{\sigma}\bar{a}_\lambda = \bar{a}^*_{\lambda-1}\bar{a}_\lambda = \bar{a}.$$

Applying Theorem 1, we see that

$$E(\mathfrak{N}^*, n, \bar{a}).$$

Contradiction.

c) $|\mathfrak{F}_\lambda| > 1$ and $\mathfrak{F}_\lambda$ is not invariant in $\mathfrak{F}_\beta$. Let $\mathfrak{B}$ be the normalizer of $\mathfrak{F}_\lambda$ in $\mathfrak{F}_\beta$. Then ([7])

$$\mathfrak{F}_\beta = \mathfrak{B}\mathfrak{G}_{\lambda-1}$$

and there exists an isomorphism φ of the group $\mathfrak{F}_\beta/\mathfrak{G}_{\lambda-1}$ onto $\mathfrak{B}/\mathfrak{B}_{\lambda-1}$, where

$$\mathfrak{B}_{\lambda-1} = \mathfrak{B} \cap \mathfrak{G}_{\lambda-1}.$$

Applying φ to the terms of the series M, we obtain the series

$$N: \quad \mathfrak{B}/\mathfrak{B}_{\lambda-1} \supseteq \cdots \supseteq \mathfrak{B}_\lambda/\mathfrak{B}_{\lambda-1} \supseteq \mathfrak{B}/\mathfrak{B}_{\lambda-1} \supseteq \cdots \supseteq \mathfrak{B}_{\lambda-1}/\mathfrak{B}_{\lambda-1}.$$

Taking into account (a) for $r = kd$, we see that M has a kd-insertion of type $a^*_{\lambda-1}$. But then, obviously, N also has a kd-insertion of the same type, and k, obviously, divides the order of the commutator subgroup of $\mathfrak{B}/\mathfrak{B}_{\lambda-1}$. Since $\mathfrak{F}_\lambda$ is not invariant in $\mathfrak{F}_\beta$, we have

$$|\mathfrak{B}| < |\mathfrak{G}|,$$

and the theorem holds for $\mathfrak{B}/\mathfrak{B}_{\lambda-1}$ with respect to $a^*_{\lambda-1}$. Therefore

$$E(\mathfrak{B}/\mathfrak{B}_{\lambda-1}, n, \mathfrak{N}/\mathfrak{B}_{\lambda-1}, a^*_{\lambda-1}).$$

Then there exists an invariant series

$$\mathfrak{N} \supseteq \cdots \supseteq \mathfrak{B}_{\lambda-1} \supseteq \mathfrak{F}_\lambda \supseteq \mathfrak{E},$$

whose indices on the segment from $\mathfrak{N}$ to $\mathfrak{B}_{\lambda-1}$ form a nonincreasing sequence σ of prime numbers. But then σ will also be the sequence of indices of the invariant series

$$\mathfrak{N}/\mathfrak{F}_\lambda \supseteq \cdots \supseteq \mathfrak{B}_{\lambda-1}/\mathfrak{F}_\lambda \supseteq \mathfrak{F}_\lambda/\mathfrak{F}_\lambda$$

on the segment from $\mathfrak{N}/\mathfrak{F}_\lambda$ to $\mathfrak{B}_{\lambda-1}/\mathfrak{F}_\lambda$. Hence, by Theorem 2, it follows that

$$E(\mathfrak{N}/\mathfrak{F}_\lambda, n, \mathfrak{N}^*/\mathfrak{F}_\lambda, \bar{a}^*_{\lambda-1}).$$

We can now apply to $\mathfrak{N}^*/\mathfrak{F}_\lambda$ the arguments of b), and, taking into account that $\mathfrak{N}^* \subseteq \mathfrak{F}_\beta$, arrive at a contradiction. For the case where a is a supersolvable sequence, it is necessary to repeat the preceding arguments, replacing throughout nilpotency by supersolvability; a kdC-sequence by a sequence of prime numbers; kdC-insertions by supersolvable insertions; a nonincreasing sequence of prime numbers by a sequence of prime numbers; references to Theorem 2 by references to the theorem of L. A. Shemetkov ([8]) (corollary).

Theorem 4. *If h is a Chunikhin nonincreasing index of the group $\mathfrak{G}$, then $\mathfrak{G}$ has a nilpotent subgroup of order h.*

Proof. As was shown above, such an index

$$h = \bar{a},$$

where a is a nilpotent sequence. Consequently, Theorem 3 can be applied to it.

Theorems 3 and 4 are analogues of Theorems 1, 4, and 5 of ([7]). For Theorems 2, 3, and 5 of ([7]), corresponding analogues can also be indicated.

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Note: Figure translations are in progress. See original paper for figures.