ON MAPPINGS IN HILBERT SPACE

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ON MAPPINGS IN HILBERT SPACE

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The principal aim of the present work is to extend the Leray–Schauder degree theory to a class of mappings M in Hilbert space. The definition of the class M is given in § 2. Mappings of this class are naturally called almost-monotone mappings. In particular, the class M includes mappings of the form $\varphi + \Phi$, where φ is a monotone mapping and Φ is a completely continuous mapping. A number of fixed-point theorems and a theorem on invariance of domain are presented.

1. Below, H always denotes a real Hilbert space. All mappings under consideration are assumed to be continuous. By I we denote the identity mapping, and by $\Phi(\Phi_i)$ completely continuous mappings.

Since one has to consider mappings of the class M that depend continuously on a parameter, it is convenient to give the definition of the class M when the domain is a closed set in the space $E = H \times R$, where R is a compact metric space. We shall usually put $R = [0, 1]$. The symbol E will always denote a space of the indicated form. Points of E will be denoted by the letter z, and points of H by the letter x. Replacing, in those definitions and propositions in which E occurs, the letter z by x and E by H, we obtain the corresponding definition or proposition for H. This makes it possible to shorten the corresponding formulations.

2. **Definition 1.** Let A be a bounded closed set in E, and let f be a mapping of A into H. We shall say that $f \in M(A)$ if the following conditions are satisfied: 1) $f(A)$ is a bounded set; 2) if A is a noncompact set, then for every noncompact set $B \subset A$

$$\limsup_{z,z' \in B} (f(z) - f(z'),\, x - x') > 0,$$

where $z = (x, t)$, $z' = (x', t')$, $x, x' \in H$, $t, t' \in R$.

Thus, if A is a bounded noncompact closed set in the Hilbert space H, then condition 2) means that for every noncompact set $B \subset A$

$$\limsup_{x,y\in B}(f(x)-f(y),\, x-y) > 0.$$

In Hilbert space, the class $M(A)$ includes mappings of the form $\varphi+\Phi$, where φ is a strictly monotone mapping, i.e. a mapping satisfying the condition

$$(\varphi(x)-\varphi(y),\, x-y) \geq \alpha(r) > 0, \qquad r = \|x-y\|.$$

A strictly monotone mapping is a mapping of the form $I-K$, where K satisfies the condition

$$(K(x)-K(y), x-y) \leq q(r)\|x-y\|^2, \qquad r=\|x-y\|, \qquad q(r)<1.$$

Definition 2. Let A be a bounded closed set in E, and let $f_1, f_2 \in M(A)$. We shall say that f_1 is homotopic to f_2 in the class M, if

there exists a homotopy $g(z,t)$, $z\in A$, $0\leq t\leq 1$, joining f_1 and f_2, such that $g(z,t)$ is a mapping of class $M(C)$, $C = A\times[0,1]$. The corresponding homotopy is called a homotopy in the class M.

Below, in Proposition 1, $L(\lambda)$ denotes the class of mappings $E\to H$ representable in the form $\lambda(z)x+\Phi(z)$, $z=(x,y)$, where $\lambda(z)$ is a positive function satisfying the condition $0<c_1<\lambda(z)<c_2<\infty$.

Proposition 1. *Let A be a bounded closed set in E. Then: 1) $M(A)$ is a convex set. 2) If $f\in M(A)$ and g_λ, g_μ are mappings of class $L(\lambda)$, then the mapping $g_\lambda f g_\mu + \Phi$ is also a mapping of class $M(A)$. 3) If $f\in M(A)$, then $f(A)$ is a closed set and, for any compact set F, the set $f^{-1}(F)\cap A$ is compact.*

3. By $T(H)$ we denote the set of all finite-dimensional subspaces of H. $T(H)$ is ordered by inclusion. If $T\in T(H)$, then by DT we denote the orthogonal complement of T, and by p_T (p_{DT}) the orthogonal projection onto T (respectively onto DT).

Let G be a bounded open set in H, $\overline{G}$ its closure, Γ the boundary of G, and f a continuous mapping $\overline{G}\to H$. For any finite-dimensional subspace T, by $G(T)$ we denote $G\cap T$, and by $\Gamma(T)$ the boundary in T of the open set $G(T)$. If $G(T)$ is nonempty, then by f_T we denote the mapping $\overline{G(T)}\to T$, defined as follows: $f_T(x) = p_T f(x)$, $x\in T\cap\overline{G}$ (p_T is the orthogonal projection onto T). If $a\in T\setminus f_T(\Gamma\cap T)$, then the degree of the mapping f_T, considered on the set $G\cap T$, at the point a is defined; we denote it by $c(f_T, a, G\cap T)$.

Let a be a point in H such that a lies outside $f(\Gamma)$. We shall say that f (considered on G) has mapping degree $c(f,a,G)$ at the point a over Z, if the following conditions are satisfied:

a) There exists a $T_0 \in T(H)$, containing a, such that for any $T \supseteq T_0$ the set $f_T(\Gamma \cap T)$ does not contain a, i.e. for any $T \supseteq T_0$ the value $c(f_T, a, G \cap T)$ is defined.

b) The values $c(f_T, a, G \cap T)$ stabilize over Z, i.e. there exists a $T_1 \in T(H)$ such that $c(f_T, a, G \cap T) = c(f, a, G)$, if $T \supseteq T_1$.

We note that if condition 1) is fulfilled, then stabilization of $c(f_T, a, G \cap T)$ takes place modulo 2, but, generally speaking, there may be no stabilization over Z; see (^{2}).

Below, to the end of the note, by mapping degree we mean mapping degree over Z.

4. For mappings of class M, the mapping degree is defined at all points lying outside the image of the boundary, and computation of the mapping degree reduces to computation of the mapping degree of a mapping of the form $I - \Phi$. More precisely, the following holds.

Proposition 2. *Let G be a bounded open set in H, Γ the boundary of G, $f : \overline{G} \to H$, and $f \in M(\overline{G})$. Then $f(\Gamma)$ is a closed set, and for any point a lying outside $f(\Gamma)$, there exists a neighborhood U in H and a $T_0 \in T(H)$ such that: 1) for any point $b \in U$ the mapping degree $c(f, b, G)$ is defined, and this value is constant on U; 2) for any $b \in U$ and any $T \supseteq T_0$, $c(f, b, G)$ coincides with the degree of the mapping $I + p_T(f - I)$ at the point b.*

In the class M, the mapping degree has all the basic properties of the mapping degree (see, for example, (^{1}), § 1): 1) additivity; 2) $a \in f(G)$, if $c(f, a, G) \neq 0$; 3) the degree $c(f, a, G)$ remains constant under a continuous deformation of the domain G and of the mapping f (in the class M), if a lies outside the image of the boundary under this deformation.

For mappings of class M the following classification theorem holds.

Proposition 3. *Let G be a bounded open set in H; Γ the boundary of G; f_1, f_2 mappings of class $M(\overline{G})$. Suppose $f_2(\Gamma) \cup f_2(\Gamma)$ does not contain the point a, $c_1 = c(f_1, a, G)$, $c_2 = c(f_2, a, G)$. The equality $c_1 = c_2$ has*

takes place if and only if there exists a homotopy $g(x, t)$ in the class M, connecting f_1 and f_2, for which $\inf \|g(x, t) - a\| > 0$, $x \in \Gamma$, $0 \leq t \leq 1$.

Let A be a bounded closed set in E, f a mapping of A into H of the class $M(A)$, $T \in T(H)$. Define the mapping $\psi[f, T]$ as follows:

$$\psi[f, T](x, t) = f(x) + tp_{DT}(x - f(x)), \qquad x \in A, \quad 0 \leq t \leq 1.$$

The mapping $\psi[f, T]$ is defined on $A \times [0, 1]$ and is a mapping of the class M. We have $\psi[f, T](x, 0) = f(x)$ and $\psi[f, T](x, 1) = x + p_T[f(x) - x]$, i.e., if t is regarded as a deformation parameter, then $\psi[f, T]$ is a deformation in the class M taking f into the mapping $I + p_T(f - I)$.

The proof of the propositions stated above is based on the following proposition:

Proposition 4. Let A be a bounded closed set in E, f a mapping of A into H of the class $M(A)$. For every point a lying outside $f(A)$, there exists a $T_0 \in T(H)$, containing the point a, and an $\varepsilon > 0$ such that, for every $T \supset T_0$, $\|p_T \psi(f, T_0)(x,t) - a\| > \varepsilon$, if $x \in T \cap A$, $0 \leq t \leq 1$.

5. Let A be a closed bounded set in E. By $M_0(A)$ we denote the set of continuous mappings $A \to H$ which is the closure, in the uniform metric, of the set $M(A)$. If A lies in H, then this is equivalent to saying that $f(A)$ is a bounded set and, for every noncompact set $B \subset A$,

$$\lim_{x,y \in B} \sup (f(x) - f(y),\ x - y) \geq 0.$$

The class M_0 includes, for example, mappings of the form $I - K - \Phi$, where $\|K(x) - K(y)\| \leq \|x - y\|$. Let G be a bounded open set in H; Γ the boundary of G; $f : \overline{G} \to H$. If $f \in M_0(\overline{G})$, then the set $f(\Gamma)$, generally speaking, may be nonclosed, and the degree of the mapping f is defined at all points lying outside $\overline{f(\Gamma)}$ in the following way.

For every point a lying outside $\overline{f(\Gamma)}$, there exist $\lambda(a) > 0$ and a neighborhood $U(a)$ such that, for every λ satisfying $0 < \lambda < \lambda(a)$, the mapping $f_\lambda = f + \lambda I$ has a degree of mapping at all points of $U(a)$, and the degree of this mapping does not depend on λ. This value is taken to be $c(f, a, G)$.

Replacing M by M_0 in Definition 2, we obtain the definition of homotopy in the class M_0. In the class M_0 the classification theorem is also valid (Proposition 3); it is only necessary, instead of $f_i(\Gamma)$, to write everywhere $\overline{f_i(\Gamma)}$.

6. For mappings of the class M the theorem on invariance of domain is valid:

Proposition 5. Let G be an open bounded set in H, f a homeomorphic mapping of G into H. If, for every closed set $A \subset G$, $f \in M(A)$, then $f(G)$ is an open set.

Below we give a number of propositions on fixed points.

Proposition 6. Let Q be an open bounded convex set in H; Γ the boundary of Q; $f : \overline{Q} \to H$. Let $f \in M(\overline{Q})$, and let a be a point of Q such that the equation $f(x) = \lambda(x - a)$ has no solutions for $x \in \Gamma$ and any $\lambda \geq 0$ (or for any $\lambda \leq 0$). Then $f(Q)$ contains the point a.

From Proposition 6 it follows immediately:

Proposition 7. Let Q be an open bounded convex set in H; Γ the boundary of Q; f such a mapping $\overline{Q} \to H$ that $I - f$ is a mapping of the class $M(\overline{Q})$. Then f has a fixed point in $\overline{Q}$, if one of the two conditions is satisfied: a) $f(\Gamma) \subseteq \overline{Q}$; b) $\overline{Q}$ is a ball and $(f(x), x) \neq (x, x)$ for $x \in \Gamma$.

The following proposition generalizes the Borsuk–Ljusternik–Schnirelman theorem:

Proposition 8. Let S be a sphere in H with center at the point O; let f be a mapping of class $M(S)$ such that the equation $f(x) = \lambda f(-x)$ has no solutions on S for any $\lambda \geq 0$. Then the degree of the mapping f with respect to the point O is odd.

Proposition 9 is a generalization of Brouwer' s theorem (3).

Proposition 9. Let f be a mapping of H into H such that the mapping $I - f$ is a mapping of class $M(A)$ for any closed set A. If, for some positive integer m, the iteration $f^m(H)$ is bounded, then f has a fixed point.

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