ON A CLASS OF GENERAL NONLOCAL ELLIPTIC PROBLEMS

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ON A CLASS OF GENERAL NONLOCAL ELLIPTIC PROBLEMS

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In the present paper we prove the Noether property of a certain class of general nonlocal problems for elliptic equations and systems (including those with discontinuous coefficients). In these problems the "boundary conditions" are given by linear differential relations connecting the values of the unknown functions and their derivatives at points of the boundary of the domain with their values on certain smooth manifolds lying inside the domain. For equations and systems of the second order, a similar problem was posed in the interesting paper (^{1}); there a method for solving this problem was also proposed, illustrated by the example of the Laplace equation with conditions of the Dirichlet type. Our method differs from that used in (^{1}). It uses the general theory of elliptic problems developed in recent years, as well as methods for solving problems with a shift ($^{2-4}$).

1. Let G be a bounded domain of n-dimensional Euclidean space; Γ its boundary; G_1 a subdomain of G with boundary γ, having no common points with Γ; $G_2 = G \setminus \overline{G_1}$. In $\overline{G_r}$ $(r = 1, 2)$ let there be given linear differential expressions

   $$L_r(x, D) = L_r\left(x, \frac{1}{i}\frac{\partial}{\partial x_1}, \dots, \frac{1}{i}\frac{\partial}{\partial x_n}\right)$$

   of order $2m_r$ with complex coefficients. Put $2m_2 + m_1 = l$. On γ there are given $2l$ linear differential expressions $B_j^r(x, D)$ $(j = 1, \dots, l;\ r = 1, 2)$, and on Γ there are l expressions of the same type. For simplicity we assume the coefficients of all the expressions considered, as well as the surfaces Γ and γ, to be infinitely smooth.

Suppose that there exists a diffeomorphism $\alpha : \Gamma \to \gamma$. Since the surfaces Γ and γ are infinitely smooth, there exists an $\varepsilon > 0$ such that, for $|t| < \varepsilon$, the mapping

$$a : x + \nu_\Gamma t \to \hat{\alpha}(x) + \nu_\gamma t$$

(ν_Γ is the unit vector of the inner normal to Γ at the point x, ν_γ is the unit vector of the normal to γ, inward with respect to G_1, at the point $\hat{\alpha}(x)$) is a diffeomorphism of some neighborhood $U(\Gamma)$ in E_n of the surface Γ onto a neighborhood $V(\gamma)$ in E_n of the surface γ. For each function $u(y)$ $(y \in V(\gamma))$ set

$$(Ju)(x) = u(a(x)) \quad (x \in \Gamma).$$

If $A(y, D_y)$ $(y \in V(\gamma))$ is an arbitrary linear differential expression with smooth coefficients, then

$$J(A(y, D_y)v(y))(x) = \hat{A}(x, D_x)(Jv)(x),$$

where $\hat{A}(x, D_x)$ $(x \in U(\Gamma))$ is a differential expression whose characteristic polynomial $\hat{A}_0(x, \xi)$ is expressed in the natural way through the characteristic polynomial $A_0(y, \eta)$ of the expression $A(y, D_y)$:

$$\hat{A}_0(x, \xi) = A_0(\alpha(x), T\xi),$$

where T is the transposed Jacobi matrix of the transformation a^{-1}.

Consider the problem:

$$L_1 u_1(x) = f_1(x) \qquad (x \in G_1), \tag{1}$$

$$L_2 u_2(x) = f_2(x) \qquad (x \in G_2), \tag{2}$$

$$B_{ju} = J(B_j^1 u_1(y) + B_j^2 u_2(y))(x) + B_j^3 u_2(x) = \varphi_j(x) \tag{3}$$

$$(x \in \Gamma,\ y = \alpha^{-1}(x) \in \gamma;\ j = 1, \dots, l).$$

We shall call problem (1)–(3) elliptic if the following conditions A, B are satisfied:

A. The differential expressions L_1, L_2 are properly elliptic in $\bar{G}_1, \bar{G}_2$, respectively. It follows from this that, for every point $x \in \Gamma$, every vector $\tau \neq 0$ tangent to Γ at the point x, and the unit normal ν to Γ at this point, the η-roots of each of the polynomials $L_{r,0}(\alpha(x), T(\tau + \eta\nu))$ $(r = 1, 2)$, $L_{2,0}(x, \tau + \eta\nu)$ are situated equally in the upper and lower half-planes. Therefore

$$L_{r,0}(\alpha(x), T(\tau + \eta\nu)) = L_r^+(\alpha(x), T(\tau + \eta\nu)) L_r^-(\alpha(x), T(\tau + \eta\nu)) \quad (r = 1, 2),$$

$$L_{2,0}(x, \tau + \eta\nu) = L_2^+(x, \tau + \eta\nu) L_2^-(x, \tau + \eta\nu),$$

where the η-roots of the polynomial L_2^+ (L_2^-) have positive (negative) imaginary parts.

B. There exist integers $\sigma_1, \dots, \sigma_l$ such that the order of the differential expression B_j^r $(r = 1, 2, 3)$ does not exceed $2m_r + \sigma_j$ $(m_3 = m_2;\ j = 1, \dots, l;\ B_j^r \equiv 0$ if $2m_r + \sigma_j < 0)$, and if $B_{j,0}^r(x, D)$ is the principal part of the expression B_j^r, which consists only of terms of order $2m_r + \sigma_j$, then at every point $x \in \Gamma$, for every

$\tau \neq 0$ tangent to Γ at the point x, and the unit normal ν to Γ at this point, the vectors

$$(B^1_{j,0}(\alpha(x), T(\tau+\eta\nu)),\ B^2_{j,0}(\alpha(x), T(\tau+\eta\nu)),\ B^3_{j,0}(x, \tau+\eta\nu))$$

$$(j = 1, \dots, l),$$

whose elements are considered as polynomials in η, are linearly independent modulo the vector

$$(L^+_1(\alpha(x), T(\tau+\eta\nu)),\ L^-_2(\alpha(x), T(\tau+\eta\nu)),\ L^+_2(x, \tau+\eta\nu)).$$

The latter means that from the simultaneous fulfillment of the equalities

$$\sum_{j=1}^{l} c_j B^1_{j,0}(\alpha(x), T(\tau+\eta\nu)) = a_1(\eta) L^+_1(\alpha(x), T(\tau+\eta\nu)),$$

$$\sum_{j=1}^{l} c_j B^2_{j,0}(\alpha(x), T(\tau+\eta\nu)) = a_2(\eta) L^-_2(\alpha(x), T(\tau+\eta\nu)),$$

$$\sum_{j=1}^{l} c_j B^3_{j,0}(x, \tau+\eta\nu) = a_3(\eta) L^+_2(x, \tau+\eta\nu),$$

where c_j are constants and $a_i(\eta)$ $(i = 1, 2, 3)$ are polynomials, it follows that $c_1 = \dots = c_l = 0$.

Theorem 1. In order that the operator

$$\Lambda\ (u_1, u_2) \to (L_1 u_1, L_2 u_2, B_1 u, \dots, B_{lu})$$

be a Noetherian operator from the space

$$W^{2m_1+k}_p(G_1) \times W^{2m_2+k}_p(G_2)$$

into the space

$$W^k_p(G_1) \times W^k_p(G_2) \times \prod_{j=1}^{l} W^{k-\sigma_j-1/p}_p(\Gamma)$$

for any nonnegative integer $k \geq \max_j\{\sigma_j\}+1$ and real $p \in (1, \infty)$, it is necessary and sufficient that problem (1)–(3) be elliptic.

Sufficiency is proved by constructing regularizers. To construct them, fix a point $x_0 \in \Gamma$ and choose a sufficiently small spherical neighborhood in E_n of this point

$U(x_0) \Subset U(\Gamma)$. Without loss of generality one may assume that $U(x_0) \cap \Gamma$ lies in the hyperplane $x_n = 0$ and that x_0 is the origin. Let us first consider problem (1)–(3)

in $(U(x_0) \cap \overline{G}_2) \cup \alpha U(x_0)$. Applying the mapping α, we reduce this problem to a problem in $U(x_0) \cap G_\alpha$ for a system of equations with unknown functions $v_1(x) = (Ju_1)(x)$, $v_2(x) = (Ju_2)(x_1, \dots, x_{n-1}, -x_n)$, $v_3(x) = u_2(x)$. Then condition A means that the latter system is elliptic in the sense of Douglis–Nirenberg, while condition B means that the corresponding boundary conditions on $\Gamma \cap U(x_0)$ satisfy the usual complementing condition. Now the construction of a regularizer is reduced to a well-studied case (see, for example, ([5],[6])).

2. Let now $L(x, D)$ be a properly elliptic linear differential expression of order $2m$ with smooth coefficients, defined in the domain $\overline{G}$. Consider the problem

$$Lu(x) = f(x), \quad x \in G, \tag{4}$$

$$C_j u = J(B_j u(y))(x) + B_j^3 u(x) = \varphi_j(x) \tag{5}$$

$$(x \in \Gamma; \quad y = \alpha^{-1} x \in \gamma; \quad j = 1, \dots, m).$$

Adding on γ $2m$ continuity conditions

$$\partial^{j-1} u_1(y)/\partial \nu_\gamma^{j-1} = \partial^{j-1} u_2(y)/\partial \nu_\gamma^{j-1} \quad (y \in \gamma; \quad j = 1, \dots, 2m),$$

we reduce this problem to problem (1)–(3); if the latter is elliptic, then we shall call problem (4)–(5) elliptic. From Theorem 1 it now follows easily that, in order that the mapping $u \to (Lu, C_1 u, \dots, C_{mu})$ be Noetherian from

$$W_p^{2m+k}(G) \text{ to } W_p^k(G) \times \prod_{j=1}^{m} W_p^{k-\sigma_j-1/p}(\Gamma),$$

it is necessary and sufficient that problem (4)–(5) be elliptic.

In particular, if conditions (5) are boundary conditions of Dirichlet type

$$J(\beta_j(y)\, \partial^{j-1} u(y)/\partial \nu_\gamma^{j-1})(x) + \partial^{j-1} u(x)/\partial \nu_\Gamma^{j-1} = \varphi_j(x) \tag{6}$$

$$(x \in \Gamma; \quad y = \alpha^{-1} x \in \gamma; \quad j = 1, \dots, m),$$

where $\beta_j(y)$ are arbitrary sufficiently smooth functions on γ, then condition B is always fulfilled, i.e. problem (4), (6) is elliptic for any properly elliptic expression $L(x, D)$.

3. It is clear that the method considered makes it possible to obtain analogous results for spaces with Hölder norms. It also makes it possible to prove the Noetherian property of a problem analogous to problem (1)–(3) also in the case when L_1, L_2 are matrix differential expressions, elliptic in the sense of Douglis–Nirenberg, and the boundary expressions satisfy a condition of type B. We do not formulate the corresponding condition here because of its cumbersome nature.

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Note: Figure translations are in progress. See original paper for figures.