ON THE SWEEP METHOD

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Abstract

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ON THE SWEEP METHOD

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A system of ordinary linear differential equations is given:

$$\mathbf{y}'(x) + P(x)\mathbf{y}(x) = \mathbf{f}(x), \qquad 0 \le x \le 1, \tag{1}$$

with summable elements $p_{ij}(x)$ of the matrix $P(x)$ and $f_i(x)$ of the vector $\mathbf{f}(x)$ $(i, j = 1, 2, \dots, n)$. We shall seek such an absolutely continuous vector $\mathbf{y}(x)$ (i.e., a vector with absolutely continuous components $y_i(x)$; everywhere below, a similar transfer of notions from scalar functions to vector and matrix functions is made) which satisfies equation (1) (almost everywhere) and the conditions

$$\int_0^1 (\mathbf{a}_i(s), \mathbf{y}(s))\, d\sigma(s) = c_i \qquad (i = 1, 2, \dots, n), \tag{2}$$

where the bracket of the form $(\mathbf{a}, \mathbf{b})$ denotes $\sum_{i=1}^n a_i b_i$, $\mathbf{a}_i(s)$ are continuous vectors, c_i are arbitrary numbers, $\sigma(s)$ is a real nondecreasing bounded function; all other quantities are assumed, generally speaking, to be complex.

In particular, for $\sigma(s) = 0, 1, 2$, respectively, when $s = 0,\ 0 < s < 1,\ s = 1$, we obtain ordinary boundary conditions of the form

$$(\mathbf{a}_i, \mathbf{y}(0)) + (\mathbf{b}_i, \mathbf{y}(1)) = c_i \qquad (i = 1, 2, \dots, n), \tag{3}$$

and, when $\sigma(s)$ is also stepwise but with more than two points of discontinuity, a multipoint problem, which we shall not write out.

As is known from the general theory of such problems (see, for example, the handbook (^{1}), and also from studies on the algorithmic development of theoretical constructions (see, for example, ($^{2-7}$)), their solution reduces to carrying out two basic procedures, for each of which computational algorithms have been developed (and continue to be improved):

(α) solution of the Cauchy problem for the corresponding homogeneous system;

(β) solution of a system of linear algebraic equations, for example, to find the initial values $\mathbf{y}(0)$ from the implicit equations (3) in the case of separated boundary conditions.

A third element, not absolutely obligatory but brought in both for the theoretical study of problems and for overcoming difficulties in their practical solution, is

(γ) solution of the adjoint equation (* denotes transposition)

$$\mathbf{z}'(x) = P^*(x)\mathbf{z}(x). \tag{4}$$

In the cited works, various combinations of these procedures are mainly considered with the aim of choosing the variant that is most optimal from the computational point of view. In this note the possibility is justified of still another variant, namely, the complete elimination of procedure (α), replacing it by (γ), placed before (β), with the introduction of a new obligatory element—**integration** (**summation**) by ⋯

parameter s with weight $d\sigma(s)$, intermediate between (γ) and (β). Such a composition of procedures can essentially be inferred from the following considerations: the direct carriers of information in conditions (2) are the coefficients $a_i(s)$, and therefore the solution of the problem should begin with the processing of this information; but they occupy, in the bilinear form $(a_i(s), y(s))$, a position algebraically conjugate to $y(s)$, and therefore must be processed in the differential problem by means of the transposed matrix, i.e., by means of equations (4).

Thus:

I. Solving on the interval $0 \le x \le 1$ system (4) with the initial condition

$$z(x)\big|_{x=s} = a_i(s), \tag{5}$$

we obtain vector-functions $z(x) \equiv z_i(x, s)$ $(i = 1, 2, \dots, n)$, continuous jointly in x, s and absolutely continuous in x.

II. Integrating $z_i(x, s)$ with respect to the parameter s with weight $d\sigma(s)$, we obtain absolutely continuous functions

$$w_i(x) = \int_0^1 z_i(x, s)\, d\sigma(s), \tag{6}$$

also satisfying equations (4).

III. The solution $y(x)$ of the posed problem (1), (2) is now determined from the algebraic system of equations

$$(w_i(x), y(x)) = c_i + \int_0^1 d\sigma(s) \int_s^x (z_i(u,s), f(u))\, du \quad (i = 1, 2, \dots, n), \tag{7}$$

whose Cramer determinant is the Wronskian $W(x)$ of the vector-functions (6).

Let us formulate the main results.

Theorem 1. *If $y(x)$ is a solution of problem (1), (2), then $y(x)$ satisfies the algebraic system of equations (7).*

Theorem 2. *If system (7) is compatible for some $x = x_0$ and has rank r, then the differential system (1), (2) is solvable and its general solution contains $n - r$ arbitrary (linearly entering) constants.*

Thus, (7) constitutes a **complete system of integrals** of problem (1), (2). Solving this system by Cramer's formulas, for $W(x) \neq 0$, we obtain the formulas of the Cauchy—Green method: this follows from the known relation between the fundamental matrices of solutions of the main and adjoint homogeneous systems of equations.

In particular, for the boundary-value problem (1), (3) we solve equation (4) with initial values $z(0) = a_i$ (forward sweep), and in parallel—with initial values $z(1) = b_i$ (backward sweep, **not dependent** on the first); we obtain the corresponding functions $z_i(x, 0)$ and $z_i(x, 1)$, and system (7) takes the form of the integrals of the boundary-value problem

$$(w_i(x), y(x)) = c_i + \int_0^x (z_i(u,0), f(u))\, du + \int_1^x (z_i(u,1), f(u))\, du,$$

where $w_i(x) = z_i(x, 0) + z_i(x, 1)$.

For the proof of both theorems, with each summand quadratic to the matrix $P = P(x),\ 0 \leq x \leq 1$, we associate its matricant

$$M(x, s; P) = E + \int_s^x P(u)\, du + \int_s^x P(u)\, du \int_s^u P(v)\, dv + \cdots \tag{8}$$

(E is the identity matrix)—a functional matrix which, as is known [8], has the following properties (in 1), 2), 3) we do not write the letter P): 1) it is absolutely continuous in x, in s, and almost everywhere $\partial M(x,s)/\partial x = P(x)M(x,s)$; 2) $M(s,s) = E$; 3) $M(x,s)M(s,t) = M(x,t)$; 4) $M(x, s; -P^*) = M^*(s, x; P)$.

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* Thus, by Theorem 1, *system* (7) *is compatible for all x*, $0 \leq x \leq 1$, *and has rank r*.

From these properties one immediately obtains the following identity for any solution $y(x)$ of equation (1): if for the matrix $P = P(x)$ of equation (1) the matricant $M(x, s; -P)$ is denoted by $A(x, s)$, then

$$y(x) = A(x, s)y(s) + \int_s^x A(x, u)f(u)\,du; \tag{9}$$

and conversely: a function of the form (9) with a prescribed vector b in place of $y(s)$ satisfies equation (1) and has $y(s) = b$.

Let us prove Theorem 1. Suppose $y(x)$ satisfies (1), (2). Write for it the identity (9), interchange x and s in it, substitute the conditions (2), and move the matrix factor from the second term of the scalar product to the first; after simple transformations we obtain

$$\left(\int_0^1 A^*(s, x)a_i(s)\,d\sigma(s), y(x)\right) = c_i + \int_0^1 d\sigma(s) \int_s^x \left(A^*(s, u)a_i(s), f(u)\right)\,du. \tag{10}$$

But by virtue of property 4) the matrix $A^*(s, x) = M^*(s, x; -P)$ is the matricant $M(x, s; P^*) = B(x, s)$ of the matrix $P^* = -(-P^*)$, and therefore the function

$$z_i(u, s) = A^*(s, u)a_i(s) = B(u, s)a_i(s) \tag{11}$$

on the basis of the corresponding identity of type (9) satisfies the differential problem (4), (5), i.e. coincides with the function denoted in the same way in item 1. Consequently, (10) coincides with (7), as was required to prove.

Let us prove Theorem 2. Suppose the numerical system

$$(w_i(x_0), y) = c_i + \int_0^1 d\sigma(s) \int_s^{x_0} \left(z_i(u, s), f(u)\right)\,du \tag{12}$$

is consistent, has rank r, and has the general solution

$$y = b = b_0 + l_1 b_1 + \dots + l_m b_m \qquad (m = n - r)$$

of the standard structure: b_0 is a particular solution of the complete system (12), b_j are linearly independent solutions of the corresponding homogeneous system, and l_j are arbitrary constants ($j = 1, 2, \dots, m$). Then the function

$$y(x) = A(x, x_0)b + \int_{x_0}^x A(x, u)f(u)\,du \tag{13}$$

by virtue of (9) satisfies equation (1) and contains m arbitrary constants (and, in an obvious way, is represented in the corresponding standard form—for solutions of differential equations). To verify condition (2), we replace x by s in (13) and substitute into the left-hand side of (2).

Repeating the arguments of the proof of Theorem 1, we obtain the number H_0, but, by (13), $y(x_0) = b$, and therefore (14), by virtue of (12), is equal to the number c_i. The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.

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