EXPANSION IN EIGENFUNCTIONS OF A NON-SELF-ADJOINT SCHRÖDINGER OPERATOR

MATHEMATICAL PHYSICS

1970

SovietRxiv

Abstract

Full Text

UDC 517.9:513.88:519:55:539.1

MATHEMATICAL PHYSICS

A. G. RAMM

EXPANSION IN EIGENFUNCTIONS OF A NON-SELF-ADJOINT SCHRÖDINGER OPERATOR

(Presented by Academician V. A. Fock on 12 VI 1969)

The paper gives a simple method of proving the theorem on expansion in eigenfunctions of a non-self-adjoint Schrödinger operator. Although the method is simple, apparently it has not been applied to the problem under consideration ([2–7]). The study of the Schrödinger operator with a complex potential is of interest, in particular, for nuclear physics ([1]). In the first part of the paper a proof is given of the theorem on expansion in eigenfunctions of a non-self-adjoint Schrödinger equation. In the second part a number of remarks are made.

I. Let $f(x)$ be a twice differentiable finite function, $\mathcal{L} = -\Delta + q(x)$, and let $q(x)$ be a complex-valued differentiable function satisfying the estimate $|q(x)| \leq C\exp(-a|x|)$, $a > 0$, $x = (x_1, x_2, x_3)$. Here and below the letter C denotes estimating constants. From the results of papers ([8a–,]) it follows that, under the assumptions made, the following assertions are true:

1. Let $R(x, y, \sqrt{\lambda})$ be the kernel of the resolvent $(\mathcal{L} - \lambda I)^{-1}$ of the operator $\mathcal{L}$. Then
   $R(x, y, \sqrt{\lambda}) = R(y, x, \sqrt{\lambda})$,

$$(\mathcal{L}_z - \lambda I)R(x, z, \sqrt{\lambda}) = \delta(x - z); \quad (\mathcal{L}_z - \lambda I)R(z, y, -\sqrt{\lambda}) = \delta(z - y). \quad (1)$$

The resolvent kernel satisfies the integral equations

$$R(x, y, k) = \frac{\exp(ik|x-y|)}{4\pi|x-y|} - \int \frac{\exp(ik|x-t|)}{4\pi|x-t|} q(t)R(t, y, k)\, dt, \quad (2)$$

$$R(x, y, k) = \frac{\exp(ik|x-y|)}{4\pi|x-y|} - \int \frac{\exp(ik|y-t|)}{4\pi|y-t|} q(t)R(x, t, k)\, dt, \quad k = \sqrt{\lambda}. \quad (3)$$

The integration is carried out over the whole three-dimensional space. For all k belonging to the half-plane $\operatorname{Im} k > -b$, where $b < a/2$, except for a finite number of points k_j, equations (2) are uniquely solvable in the space $C(E_3, \exp[-\frac{1}{2}a|x|])$. The kernel $R(x, y, k)$ is analytic in k everywhere for $\operatorname{Im} k > -b$, except for the mentioned points k_j, which are poles of the kernel.

2. The poles k_j, $1 \le j \le p$, lying in the region $\operatorname{Im} k > 0$, are eigenvalues of the operator $\mathcal{L}$. The poles lying in the region $\operatorname{Im} k \le 0$ are not eigenvalues of the operator $\mathcal{L}$, since the corresponding eigenfunctions, generally speaking, are not square integrable. These poles are called non-spectral singularities (quasi-eigenvalues). Denote by ν_j, $1 \le j \le q$ (μ_j; $0 \le j \le s$), the poles k_j lying on the negative (nonnegative) real half-axis. As examples show, the point $k = 0$, even in the case of a real potential, may be either an eigenvalue or a quasi-eigenvalue of the operator $\mathcal{L}$.

3. The estimate holds

$$\max_{x \subset E_3} \exp\left(-\frac{1}{2}a|x|\right) \left| \int R(x, y, k) f(y)\, dy \right| \le \frac{C}{1 + |k|},$$

$$\operatorname{Im} k > -b, \quad |k - k_j| \ge \delta > 0; \quad C = C(f, \delta).$$

4. The correct formulas are:

$$R(x, y, k) = \frac{\exp(ik|x|)}{4\pi|x|}\, u(y; k, n)(1 + o(1)), \quad |x| \to \infty, \ \frac{x}{|x|} = n; \tag{4a}$$

$$R(x, y, k) = \frac{\exp(ik|y|)}{4\pi|y|}\, u(x; k, n)(1 + o(1)), \quad |y| \to \infty, \ \frac{1}{|y|} = n. \tag{4b}$$

5. The solutions of the scattering problem are determined from the equation

$$u(x; k, n) = \exp[-ik(n, x)] - \int \frac{\exp(ik|x - t|)}{4\pi|x - t|}\, q(t)\, u(t; k, n)\, dt, \tag{5}$$

which is uniquely solvable in $C(E_3, \exp[-\frac{1}{2}a|x|])$ for $k \ne k_j$, $\operatorname{Im} k > -b$. The solution of equation (5) is analytic in k for $k \ne k_j$, $\operatorname{Im} k > -b$.

To prove the expansion theorem, we integrate the easily verified identity $f/k^2 = -R(k)f + R(k)\mathcal{L}f/k^2$ over the contour $C_N:\ |k| = N,\ 0 \le \arg k \le \pi$, after first multiplying both sides of it by k. Here $R(k)$ denotes the operator $(\mathcal{L} - k^2 I)^{-1}$ with kernel $R(x, y, k)$. We obtain

$$f = -\frac{1}{i\pi} \int_{C_N} R(k) f k\, dk + \frac{1}{i\pi} \int_{C_N} \frac{R(k)\mathcal{L}f}{k}\, dk = J_1 + J_2. \tag{6}$$

Let $\mathcal{L}f$ be a twice differentiable function. Then, by assertion 3, the equality $\lim_{N\to\infty} J_2 = 0$ holds, where the limit is understood in the sense of convergence in $C(E_3, \exp[-\frac{1}{2}a|x|])$, and therefore is attained uniformly in every finite domain $D \subset E_3$. We transform J_1, using assertion 1 and assuming in addition that $\mu_0 \neq 0$ (we shall get rid of this assumption below). Applying Cauchy' s theorem, we obtain

$$\lim_{N\to\infty}\left[-\frac{1}{\pi}\int_{C_N} R(k)fk\,dk\right] = \frac{1}{i\pi}\int_{\gamma} R(k)fk\,dk + \sum_{j=1}^{p+q} -\frac{1}{i\pi}\oint_{\gamma_j} R(k)fk\,dk. \quad (7)$$

Here γ_j is a small circle with center at the point k_j; the sum extends over the poles lying in the region $\operatorname{Im} k > 0$, and over the poles ν_j. Note that [(4)]

$$-\frac{1}{i\pi}\oint_{\gamma_j} R(k)fk\,dk = P_{jf},$$

where P_j is the projector onto the root subspace of the operator $\mathcal{L}$ corresponding to the number k_j^2.* The contour γ goes along the real axis of the k-plane, has its origin at the center of symmetry (this is possible, since $\mu_j \neq 0$), encircles the points $\nu_j,\ 1 \le j \le q$, and $-\mu_j,\ 0 \le j \le s$, by small semicircles lying in the region $\operatorname{Im} k < 0$, and the points $\mu_j,\ 0 \le j \le s$, and $-\nu_j,\ 1 \le j \le q$, by small semicircles lying in the region $\operatorname{Im} k > 0$. Denote by γ_+ the part of the contour γ lying in the region $\operatorname{Re} k \ge 0$. We have

$$\frac{1}{i\pi}\int_{\gamma} R(k)fk\,dk = \frac{1}{i\pi}\int_{\gamma_+} dk\,k\,[R(k) - R(-k)]f. \quad (8)$$

To obtain the expansion theorem, it remains to express the difference $R(k) - R(-k)$ in terms of solutions of the scattering problem $(8a, 9)$, using equation (2), Green' s formula, and the asymptotics (4); we obtain:

$$R(x,y,k) - R(x,y,-k) =$$

$$= \lim_{r\to\infty}\int_{|z|\le r}\{R(z,y,-k)\Delta_z R(x,z,k) - R(x,z,k)\Delta_z R(z,y,-k)\}\,dz =$$

$$= \lim_{r\to\infty}\int_{|z|=r}\left\{R(z,y,-k)\frac{\partial R(x,z,k)}{\partial |z|} - R(x,z,k)\frac{\partial R(z,y,-k)}{\partial |z|}\right\}ds =$$

$$= \frac{ik}{8\pi^2}\int_S u(x;k,n)u(y;-k,n)\,dn. \quad (9)$$

* For $\operatorname{Im} k_j > 0$.

Here S is the unit sphere of three-dimensional space, dn is its surface-area element. Introduce the notation

$$\hat{f}(k,n) = \frac{1}{(2\pi)^{3/2}} \int f(y) u(y; -k, n)\, dy. \tag{10}$$

Then formula (6), taking into account the equalities (7)–(10), takes the form

$$f(x) = \frac{1}{(2\pi)^{3/2}} \int_{\gamma_+} \int_S \hat{f}(k,n) u(x;k,n) k^2\, dk\, dn + \sum_{j=1}^{p+q} P_j f. \tag{11}$$

Formulas (10), (11) constitute the expansion theorem in eigenfunctions of the operator $\mathcal{L}$. If all poles k_j lying in the region $\operatorname{Im} k > -b$, $b > 0$, are non-real, then the contour γ_+ coincides with the half-axis $[0, \infty)$, $q = 0$, and formula (11) takes a form analogous to formula (16) in (8^a), where the expansion theorem for the self-adjoint operator $\mathcal{L}$ was proved.

Let us introduce formulas analogous to formulas (10), (11), without assuming that $\mu_0 \neq 0$. If μ_0 is a pole of first order, then the function $kR(x,y,k)$ is analytic in a neighborhood of the point $k = 0$, and the preceding arguments remain valid without any changes. If $\mu_0 = 0$ is a pole of second order of the kernel $R(x,y,k)$, then the function $kR(x,y,k)$ has a simple pole at the point $k = 0$. If we take a contour γ_1 coinciding with the contour γ described after formula (7) everywhere outside a small neighborhood of zero and going around zero by a small semicircle of radius $\varepsilon > 0$, lying in the region $\operatorname{Im} k \geq 0$, then the arguments given after formula (7) remain valid, but on the right-hand side of formula (8) there is added the contribution from the small semicircle C_ε, going around the point $k = 0$, equal to $-\int a(x,y) f(y)\, dy$, where $a(x,y)$ is the residue of $kR(x,y,k)$ at the point $k = 0$. Therefore, in the case under consideration, as $\varepsilon \to 0$ we obtain

$$\lim_{\varepsilon \to 0} \frac{1}{i\pi} \int_{\gamma_1} R(k) f k\, dk = \frac{1}{i\pi} \int_{\gamma_+} [R(k) - R(-k)] f k\, dk - \int a(x,y) f(y)\, dy. \tag{12}$$

Consequently, formula (11) takes the form

$$f(x) = \frac{1}{(2\pi)^{3/2}} \int_{\gamma_+} \int_S \hat{f}(k,n) u(x;k,n) k^2\, dk\, dn + \sum_{j=1}^{p+q} P_j f - \int a(x,y) f\, dy, \tag{13}$$

where the notation of (11) is retained. Let us formulate the results obtained.

Theorem 1. Suppose that the assumptions made at the beginning of Sec. I are fulfilled, $\mathcal{L}f$ is a twice differentiable function, and the point $k = 0$ is a point of analyticity or a pole of first order of the kernel $R(x, y, k)$ of the resolvent of the operator $\mathcal{L}$ (this kernel is determined from the integral equation (2) or (3)). Then the inversion formulas (10), (11) are valid. If the point $k = 0$ is a pole of second order of the kernel $R(x, y, k)$, then the inversion formula takes the form (13), where

$$a(x, y) = \operatorname*{Res}_{k=0} kR(x, y, k).$$

Consider the case when the point $k = 0$ is a pole of the kernel of order $n + 1$, $n > 1$. Take as γ in formula (7) the contour γ_1. If $n > 1$, then the quantity

$$\int_{C_\varepsilon} R(k) f k \, dk$$

does not have a finite limit as $\varepsilon \to 0$. Therefore, instead of formula (12) we obtain the equality

$$\frac{1}{i\pi} \int_{\gamma_1} R(k) f k \, dk = \frac{1}{i\pi} \int_{\gamma_+^\varepsilon} dk \, k \, [R(k) - R(-k)] f + \int_{C_\varepsilon} R(k) f k \, dk. \qquad (14)$$

Here γ_+^ε is the part of the contour γ_+ located in the region $\operatorname{Re} k > \varepsilon$. Instead of formula (13), taking equality (14) into account, we obtain:

$$f(x) = \frac{1}{(2\pi)^{3/2}} \int_{\gamma_+^\varepsilon} \int_S \hat{f}(k, n) u(x; k, n) k^2 \, dk \, dn + \sum_{j=1}^{p+q} P_j f +$$
$$+ \int_{\dot{C}_\varepsilon} dk \, k \int R(x, y, k) f(y) \, dy. \qquad (15)$$

(15) plays the role of an inversion formula in the general case under consideration.

II. Let us consider some remarks connected with the method of proof of the eigenfunction expansion theorem given in Sec. I. The following points are essential for the proof presented.

1) The kernel $R(x, y, \sqrt{\lambda})$ of the resolvent of the differential operator admits an analytic continuation into the half-plane $\operatorname{Im} k > -b$, $b > 0$, $k = \sqrt{\lambda}$, and this continuation is meromorphic and has a finite number of poles in the domain $\operatorname{Im} k > -\varepsilon$, where $\varepsilon > 0$ is an arbitrarily small but fixed

number.

2) The quantity $R(k)f$ tends to zero as $|k| \to \infty$, $0 \leq \arg k \leq \pi$. Instead of assertion 3 of Sec. 1, it would have been sufficient to have the relation

$$\lim_{\substack{k\to\infty \\ 0\leq \arg k\leq\pi}} \int R(x,y,k)f(y)\,dy = 0$$

uniformly with respect to x, ranging over any bounded domain. Therefore the assumption that the function $\mathcal{L}f$ is twice differentiable can be weakened.
3) For the kernel there hold the asymptotic formulas (4), which make it possible to express the difference $R(k) - R(-k)$ through solutions of the scattering problem.

Since in two-dimensional problems the analytic continuation of the kernel $R(x,y,k)$ is carried out onto a plane with a cut beginning at the point $k = 0$ ([8v, g, zh]), in order to carry out the scheme of the proof of the expansion theorem given in Sec. I it is necessary to make sure that the point $k = 0$ is not an accumulation point of the poles of the kernel. In a number of cases this was done in ([8zh]).

The choice of the contour γ in formula (7) is not uniquely possible. For example, instead of γ one could take a contour $\tilde{\gamma}$, symmetric to the contour γ with respect to the real axis. In this case, for example, in formula (11) the summation would be carried out up to the index $p + s$, and the contour γ_+ would have to be replaced by the contour $\tilde{\gamma}_+$. It is desirable to choose the contour γ so that the left-hand side of equality (8) can be transformed into an integral of the quantity $[R(k) - R(-k)]f$, which can be expressed through solutions of the scattering problem. If the kernel $R(x,y,k)$ has no poles on the real axis, then the contour γ_+ in formula (11) coincides with the half-axis $[0,\infty)$.

In conclusion we note that the expansion theorem for the self-adjoint operator $\mathcal{L}$, considered in the whole space, was obtained in ([9]), and in domains with an infinite boundary—in ([8a]). In ([6]) a certain expansion theorem was announced for the non-self-adjoint operator $\mathcal{L}$, considered in three-dimensional space, obtained with the aid of an expansion in eigenfunctions of an infinite system of ordinary differential equations.

By the method given in the present paper one can prove the eigenfunction expansion theorem for the Schrödinger operator of the Dirichlet boundary-value problem in the exterior of a bounded domain with a sufficiently smooth star-shaped boundary and inside an angular domain.

Leningrad Institute
of Precision Mechanics and Optics

Received
4 VI 1969

CITED LITERATURE

1. A. S. Davydov, *Theory of the Atomic Nucleus*, Moscow, 1958.

2. M. V. Keldysh, DAN, 77, No. 1, 11 (1951).

3. M. V. Keldysh, V. B. Lidskii, Proceedings of the IV All-Union Mathematical Congress, 1, 1963, p. 101.

4. I. Ts. Gokhberg, M. G. Krein, *Introduction to the Theory of Linear Non-Self-Adjoint Operators*, "Nauka," 1965.

5. M. A. Naimark, UMN, 11, No. 6, 183 (1956).

6. M. G. Gasymov, DAN, 179, No. 3, 518 (1968).

7. R. M. Martirosyan, Izv. AN SSSR, ser. matem., 24, No. 6, 897 (1960).

8. A. G. Ramm, a) DAN, 153, No. 2, 282 (1963); Mat. sbornik, 66, No. 3, 321 (1965); b) DAN, 166, No. 6, 1319 (1966); v) UMN, 19, No. 5, 192 (1964); g) Dokl. AN AzerbSSR, 21, No. 4, 3 (1965); d) DAN, 163, No. 3, 584 (1965); e) Izv. vyssh. uchebn. zaved., Matematika, No. 1, 124 (1966); zh) Izv. AN ArmSSR, seriya matem., 3, No. 6, 443 (1968).

9. A. Ya. Povzner, DAN, 104, No. 3, 360 (1955).

Note: Figure translations are in progress. See original paper for figures.