GROUP ALGEBRAS OF ANALYTIC FUNCTIONALS AND THEIR REPRESENTATIONS

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GROUP ALGEBRAS OF ANALYTIC FUNCTIONALS AND THEIR REPRESENTATIONS

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This note describes a special group algebra of a complex Lie group—the algebra of analytic functionals. The results obtained on the connection of the algebra of analytic functionals with other group algebras and on representations of these algebras are useful in the study of nonunitary representations of Lie groups. In its ideas the proposed work is closely connected with the work of P. K. Rashevskii ([1]).

1. Let G be a complex analytic Lie group, $\mathcal{H}(G)$ the linear space* of functions holomorphic on G, endowed with the topology of uniform convergence on compact sets. The space of linear continuous functionals on $\mathcal{H}(G)$—such functionals will be called **analytic** (cf. ([2]))—endowed with the strong topology of the conjugate space, will be denoted by $\mathcal{A}(G)$. If $f \in \mathcal{A}(G)$, $\varphi \in \mathcal{H}(G)$, denote by $\langle f, \varphi \rangle$ the value of the functional f on the function φ. For fixed $g \in G$ the function $(L_g\varphi)(x) = \varphi(gx)$ is holomorphic in x, so that $L_g\varphi \in \mathcal{H}(G)$. Denote by $f \circ \varphi$ the function which at the point $g \in G$ takes the value $\langle f, L_g\varphi \rangle$; one can show that $f \circ \varphi \in \mathcal{H}(G)$. We shall call the **convolution** of functionals f_1 and f_2, belonging to the space $\mathcal{A}(G)$, the functional $f_1 * f_2$, whose value on any function $\varphi \in \mathcal{H}(G)$ is equal to $\langle f_1, f_2 \circ \varphi \rangle$.

Theorem 1. *The space $\mathcal{A}(G)$ is an associative algebra with respect to convolution. The mapping $\mathcal{A}(G) \times \mathcal{A}(G) \to \mathcal{A}(G)$ defined by convolution is continuous.*

In the case when the group G is a complex vector space, the algebra of analytic functionals was defined and studied by L. Ehrenpreis in connection with the theory of functions periodic in the mean (see ([3])).

2. Let G be a connected complex Lie group with algebra $\mathfrak{G}$, and let $\mathcal{F}(\mathfrak{G})$ be the associative hyperenveloping algebra of the algebra $\mathfrak{G}$ in the sense of the work of P. K. Rashevskii ([1]). The group G is called a Stein group (see ([4])) if it is imbedded as a closed complex analytic submanifold in a complex Euclidean space. We note that every complex simply connected group is a Stein group, as is every connected complex semisimple group.

Proposition 1. *There exists a canonical homomorphism $\alpha : \mathcal{F}(\mathfrak{G}) \to \mathcal{A}(G)$; the mapping α is continuous and its image is dense in $\mathcal{A}(G)$. If G is a Stein group, then α is a monomorphism.*

3. Let G be an arbitrary Lie group, $C(G)$ the space of continuous functions endowed with the topology of uniform convergence on compact sets, and $C^\infty(G)$ the space of infinitely differentiable functions on G, endowed with the topology of uniform convergence with derivatives on compact sets. Denote by $\mathcal{M}(G)$ and $\mathcal{D}(G)$ the spaces, endowed with the strong topology, conjugate respectively to $C(G)$ and $C^\infty(G)$. Pro-

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* All linear spaces are considered over the field of complex numbers.

the space $\mathcal{M}(G)$ consists of all measures on G with compact supports, and $\mathcal{D}(G)$ of all generalized functions with compact supports. It is well known that $\mathcal{M}(G)$ and $\mathcal{D}(G)$ are associative algebras with respect to convolution. The operator $i : \mathcal{M}(G) \to \mathcal{D}(G)$, conjugate to the operator of the natural embedding $C^\infty(G) \to C(G)$, is a continuous monomorphism whose image is dense in $\mathcal{D}(G)$.

4. Let G be a complex Lie group, and let β be the mapping $\mathcal{M}(G) \to \mathcal{A}(G)$ conjugate to the natural embedding $\mathcal{H}(G) \to C(G)$. Analogously one defines the mapping $\tilde{\beta} : \mathcal{D}(G) \to \mathcal{A}(G)$.

Proposition 2. The mappings β and $\tilde{\beta}$ are continuous epimorphisms. The following diagram is commutative:

$$\begin{array}{ccc} \mathcal{M}(G) & \overset{i}{\rightarrow} & \mathcal{D}(G) \\ & \searrow_{\beta} \quad \swarrow^{\tilde{\beta}} & \\ & \mathcal{A}(G) & \end{array}$$

5. Let G_C be a complex Lie group with Lie algebra $\mathfrak{G}_C$, and let G_R be the real form of the group G_C. We construct the mapping $\mathcal{H}(G_C) \to C^\infty(G_R)$ which assigns to each function in $\mathcal{H}(G_C)$ its restriction to the subgroup G_R; let γ be the conjugate mapping $\mathcal{D}(G_R) \to \mathcal{A}(G_C)$. It is known that the subalgebra in $\mathcal{D}(G_R)$ consisting of generalized functions whose support is concentrated at the identity of the group G_R is isomorphic to the universal enveloping algebra $\mathcal{U}(\mathfrak{G}_C)$ of the Lie algebra $\mathfrak{G}_C$. Let ξ be the indicated isomorphism $\mathcal{U}(\mathfrak{G}_C) \to \mathcal{D}(G_R)$, η the natural embedding $\mathcal{U}(\mathfrak{G}_C) \to \mathcal{F}(\mathfrak{G}_C)$ (see (1)), and j the natural (continuous) embedding $\mathcal{M}(G_R) \to \mathcal{M}(G_C)$.

Proposition 3. The following diagram is commutative:

$$\begin{array}{ccccc} \mathcal{F}(\mathfrak{G}_C) & \xrightarrow{\alpha} & \mathcal{A}(G_C) & \xleftarrow{\beta} & \mathcal{M}(G_C) \\ \uparrow \eta & & \uparrow \gamma & & \downarrow j \\ \mathcal{U}(\mathfrak{G}_C) & \xrightarrow{\xi} & \mathcal{D}(G_R) & \xleftarrow{i} & \mathcal{M}(G_R) \end{array}$$

Proposition 4. The mapping γ is a continuous homomorphism whose image is dense in $\mathcal{A}(G_C)$. If G_C can be embedded as a closed complex analytic submanifold in the complex Euclidean space $C^n = R^n + iR^n$ in such a way that $G_R = G_C \cap R^n$, then γ is a monomorphism.

6. Let V be a locally convex topological vector space, V' the space conjugate to V, and $\mathcal{S}(V)$ the algebra of weakly continuous operators on the space V, endowed with the weak operator topology. A **continuous representation** of the algebra $\mathcal{A}(G)$ in the space V will mean a continuous homomorphism of this algebra into the algebra $\mathcal{S}(V)$.

Let δ_g be such an element of $\mathcal{A}(G)$ that $\langle \delta_g, \varphi \rangle = \varphi(g)$ for every function $\varphi \in \mathcal{H}(G)$. If T is a continuous representation of the algebra $\mathcal{A}(G)$, then the correspondence $g \mapsto \widetilde{T}_g = T(\delta_g)$ is a representation of the group G, and we shall say that the representation $\widetilde{T}$ of the group G extends to the representation T of the algebra $\mathcal{A}(G)$.

A representation $\widetilde{T}$ of the group G in the space V will be called **holomorphic** if every matrix element $\varphi(g) = (l, \widetilde{T}_{gx})$ is a holomorphic function on G and, as an element of the space $\mathcal{H}(G)$, depends weakly continuously on l and x (here $x \in V$, $l \in V'$, and $(l, \widetilde{T}_{gx})$ is the value of the functional l on the element $\widetilde{T}_{gx} \in V$).

Proposition 5. A representation of the group G is holomorphic if and only if it extends (always in a unique way) to a continuous representation of the algebra $\mathcal{A}(G)$.

7. It is not difficult to verify that the right-regular representation of a complex group G in the space $\mathcal{H}(G)$ extends to a continuous representation R of the algebra $\mathcal{A}(G)$, where $R(f) : \varphi \mapsto f \circ \varphi$, $f \in \mathcal{A}(G)$, $\varphi \in \mathcal{H}(G)$. Closed subspaces in $\mathcal{H}(G)$ and in $\mathcal{A}(G)$ will be called **planes**. To each plane $E \subset \mathcal{H}(G)$ we assign the plane $E' \subset \mathcal{A}(G)$, consisting of all such elements $f \in \mathcal{A}(G)$ that $\langle f, \varphi \rangle =$

-0 for any function $\varphi \in \mathcal{H}(G)$. The space $\mathcal{H}(G)$ is reflexive; therefore the correspondence $E \to E'$ between planes in $\mathcal{H}(G)$ and $\mathcal{A}(G)$ is one-to-one.

Theorem 2. *In order that a plane $E \subset \mathcal{H}(G)$ be invariant with respect to the right-regular representation of the group G or of the algebra $\mathcal{A}(G)$, it is necessary and sufficient that the plane E' be a right ideal. In particular, the one-to-one correspondence established between irreducible subrepresentations of the regular representation and closed maximal (among closed) right ideals in $\mathcal{A}(G)$ holds.*

In the case when G is a complex vector space, Theorem 2 was proved in the work of L. Ehrenpreis ([3]). For associative superenveloping algebras of Lie algebras an analogous theorem was proved in the work of P. K. Rashevskii ([1]).

It is easy to verify that if I is a closed right ideal and

$$I_f = \{g \in \mathcal{A}(G) : \ g * f \in I\},$$

then the kernel of the representation corresponding to the ideal I coincides with the intersection of all closed right ideals I_f (f runs through the whole algebra $\mathcal{A}(G)$).

8. Representations of the algebra $\mathcal{D}(G)$ and of the algebra $\mathcal{M}(G)$, endowed with the weak topology of the space conjugate to $C(G)$, are also connected with representations of the group G, whose matrix elements are respectively infinitely differentiable or continuous, just as representations of the algebra of analytic functionals of a complex group are connected with holomorphic representations of this group. For the algebras $\mathcal{D}(G)$ and $\mathcal{M}(G)$, analogues of Theorem 2 and Proposition 6 are also valid. It is reasonable to study irreducible representations of group algebras up to equivalence in the sense of Fell (see ([5])), declaring representations equivalent if their kernels coincide. It is well known that unitary representations of a locally compact group G can be regarded as representations of the group C^*-algebra; if G is a group of type I, then irreducible representations of this algebra have the same kernels precisely when the corresponding representations of the group G are unitarily equivalent. Thus, one may consider several parallel (and interconnected) theories of group representations by considering representations of various group algebras. An idea of this kind was expressed several years ago by D. P. Zhelobenko.

9. If the complex Lie group G is semisimple, then irreducible representations of the algebra $\mathcal{A}(G)$ are finite-dimensional; such representations are equivalent in the sense of Fell if and only if they are equivalent in the usual sense. Any maximal ideal in $\mathcal{A}(G)$ (right, left, or two-sided) has finite codimension; it is easy to describe how any ideal in $\mathcal{A}(G)$ is arranged. All this is readily derived from the fact that the group G possesses a compact real form. If, however, the group G, for example, is nilpotent, then the algebra $\mathcal{A}(G)$, generally speaking, has maximal ideals of infinite codimension and infinite-dimensional irreducible representations. For a description of completely irreducible representations of the algebra of analytic functionals of a nilpotent group see ([6]). We note that if the group G is the complex vector space $\mathbb{C}^n$, then the problem of describing maximal ideals in the algebra $\mathcal{A}(\mathbb{C}^n)$ is well known (see ([3])) and for $n > 1$ has still not been solved.

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Note: Figure translations are in progress. See original paper for figures.