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Abstract

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ON THE CLASSIFICATION OF THE CURVATURE TENSOR BY MEANS OF PRINCIPAL CONGRUENCES

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1. The algebraic classification of the curvature tensor of a Riemannian space V_4 according to Petrov types ([1]) is carried out locally with the use of a canonical orthonormal frame. If, however, only one of the canonical or principal vectors of the tensor is used for the purposes of its classification, then the latter can be constructed not only at a point, but also in a finite region of V_4. In this connection, the splitting of the tensor being classified under the choice of a reference system is used essentially.

A reference system (r.s.) is a congruence of timelike lines ([2]). It is uniquely determined by the field of a unit tangent vector $u^\alpha(x^\mu) = dx^\alpha/ds$ (s is the arc length of the curve). With the aid of this vector the curvature tensor is split into 3 chronometrically invariant tensors (chi-tensors) of incomplete rank ([3,10]):

$$X_{\mu\nu} = R_{\mu\alpha\nu\beta}u^\alpha u^\beta, \quad Y_{\mu\nu} = {}^*R_{\mu\alpha\nu\beta}u^\alpha u^\beta, \quad Z_{\mu\nu} = {}^*R^*_{\mu\alpha\nu\beta}u^\alpha u^\beta. \tag{1}$$

Here

$${}^*R_{\mu\alpha\nu\beta} = \frac{1}{2}\eta_{\mu\alpha\sigma\rho}R^{\sigma\rho}_{\nu\beta}, \quad {}^*R^*_{\mu\alpha\nu\beta} = \frac{1}{4}\eta_{\mu\alpha\sigma\rho}R^{\sigma\rho\lambda\tau}\eta_{\lambda\tau\nu\beta},$$

(Greek indices run over the values $0, 1, 2, 3$, and Latin over $1, 2, 3$), $\eta_{\mu\alpha\nu\beta}$ is the volume tensor.

The Weyl tensor used for the Petrov classification is then split into 2 chi-tensors $U_{\mu\nu} = C_{\mu\alpha\nu\beta}u^\alpha u^\beta$, $V_{\mu\nu} = {}^*C_{\mu\alpha\nu\beta}u^\alpha u^\beta$. Both these tensors can be simultaneously brought to canonical form only in the case when the splitting vector u^μ coincides with a vector of the canonical orthonormal frame.

We shall call a reference system canonical with respect to the Weyl tensor one in which its splitting tensors can be simultaneously brought to canonical form by internal coordinate transformations $x^i = x^{i'}(x^i)$.

If the Weyl tensor belongs to Petrov type I or D, then the canonical congruence (canonical r.s.) coincides with the principal congruence generated by the principal vector of Riemann [4]. In this case $U=(U_{\mu\nu})$ and $V=(V_{\mu\nu})$ are matrices of simple structure, i.e. $UV-VU=0$, or

$$A_{\mu\nu\alpha\beta\gamma\delta}u^\alpha u^\beta u^\gamma u^\delta=0,\quad A_{\mu\nu\alpha\beta\gamma\delta}\overset{\text{def}}{=}\frac{1}{2}\left(C_{\mu\alpha\rho}{}^{*}C_{\nu\gamma\sigma\delta}-C_{\nu\alpha\beta}{}^{*}C_{\mu\gamma\sigma\delta}\right)g^{\rho\sigma}\equiv C^{\rho}{}_{\cdot\beta\alpha[\mu}{}^{*}C_{\nu]\gamma\delta\rho}. \tag{2}$$

In the canonical form of the Weyl tensor of types II, N, and III there enters a Lorentz-rotation parameter different from zero (in any frame), say σ, in the plane x^0x', which becomes 1 under a rotation through the angle φ [5]: $\sigma e^{i\varphi}=1$. Obviously, $\sigma\to 0$ as $\varphi\to -i\infty$. At the same time $u^\mu\to l^\mu$ (l^μ is the isotropic principal vector of Riemann). Thus, the canonical forms of the Weyl tensors of the three types (with identical elementary bases) coincide if, as the splitting vector, one takes only the (first, i.e. isotropic or timelike) principal vector of Riemann. Equation (2) is correspondingly applicable to gravitational fields of all types that differ by the norm of the (first) vector which is its solution.

If all elementary bases vanish (we shall say that in this case the Weyl tensor belongs to class 3; types II and D are assigned to class 2, and type I to class 1), then equation (2) is satisfied trivially, and types N and III must be distinguished with the aid of the well-known equations

$$C_{\mu\alpha\nu\beta}l^\alpha=0; \tag{a}$$

$$C_{\mu\alpha\nu\beta}l^\alpha\neq 0,\qquad C_{\mu\alpha\nu\beta}l^\alpha l^\beta=0. \tag{b}$$

(3)

Thus, the classification in a region looks as follows. Finding the scalar invariants in the usual way and setting them equal to zero, we divide the region of space under consideration into subregions, in each of which the Weyl tensor belongs to the 1st, 2nd, or 3rd class. Solving equation (2) or (3), we find the (first) principal congruence of the Weyl tensor. The hypersurface of degeneration S of the principal congruence will divide the region into subregions of types D and II, while the boundary of double degeneration (all principal vectors merge with the isotropic one) will divide the region into subregions of types N and III.

If on the hypersurface S the principal congruence has a point of inflection, touching there the local isotropic cones, then, correspondingly, the space will be degenerate only on S. If radiation persists and propagates in empty space, then the corresponding congruence will, as is known, be geodesic: otherwise there would appear scalars different from zero (the curvature of the congruence).

All characteristics of the principal congruence (its 3-curvatures, the existence of normal congruences of hypersurfaces, etc.) are invariant characteristics of the region and make it possible in an obvious way to refine the classification of the curvature-tensor field. Example*: the first curvature χ_1 of the principal congruence of the Schwarzschild field becomes infinite on the singular sphere, thus dividing the space of type D into two parts, while the sphere itself has type II; this is an inotype surface.

The physical character of the singularity on the sphere is also evident from the fact that the density of gravitational energy (we define it as $-\chi_1^2$; this is the Einstein analogue of the density of Newtonian gravitational energy) of the canonical s.r. also becomes infinite on the sphere.

The principal congruence simplifies the geometric description of the region V_4, but for the physical interpretation it is necessary to introduce a uniquely determined canonical s.r. (let us note, incidentally, a circumstance characteristic of general relativity, consisting in the existence of a distinguished reference system; this is caused by the inhomogeneity of the curvature of space).

Introducing a locally canonical (at the point P), semigeodesic s.r. in the region and using the linear covariant approximation for the case of vacuum (i.e. $X_{\mu\nu}(=-Z_{\mu\nu})$ and $Y_{\mu\nu}$ depend only on the tensor of deformation velocities (?) $D_{ik} = \partial g_{ik}/\partial x^0$ and its derivatives, and in a neighborhood of P we take $D_{iD_{ik}} = 0$, since $D_{ik}(P) = 0$), it is not difficult to obtain that real stationary curvatures correspond to dilatations, while imaginary ones correspond to shifts of the mentioned s.r. Type N corresponds, in particular, to a transverse-transverse wave of deformation and curvature of the s.r., and type III to a longitudinal-transverse shear wave.

2. In order to obtain a detailed classification of the Riemann tensor, and moreover in a form comparable with its classification by the Weyl tensor, we use the following decomposition of the curvature tensor into irreducible components ([6]):

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* Considered jointly with V. E. Yakimov and L. P. Grishchuk.

$$R_{\mu\alpha\nu\beta} = C_{\mu\alpha\nu\beta} + E_{\mu\alpha\nu\beta} + G_{\mu\alpha\nu\beta},$$

$$\begin{aligned} C_{\mu\alpha\nu\beta} &= R_{\mu\alpha\nu\beta} + \frac{1}{2}\left(R_{\mu\nu}g_{\alpha\beta} + R_{\alpha\beta}g_{\mu\nu} - R_{\mu\beta}g_{\nu\alpha} - R_{\nu\alpha}g_{\mu\beta}\right) - \frac{1}{6}Rg_{\mu\alpha\nu\beta} \\ &= -{}^*R^*_{\mu\alpha\nu\beta} - \frac{1}{2}\left(R_{\mu\nu}g_{\alpha\beta} + R_{\alpha\beta}g_{\mu\nu} - R_{\mu\beta}g_{\nu\alpha} - R_{\nu\alpha}g_{\mu\beta}\right) + \frac{1}{3}Rg_{\mu\alpha\nu\beta}, \end{aligned}$$

$$\begin{aligned} E_{\mu\alpha\nu\beta} &= -\frac{1}{2}\left(R_{\mu\nu}g_{\alpha\beta} + R_{\alpha\beta}g_{\mu\nu} - R_{\mu\beta}g_{\nu\alpha} - R_{\nu\alpha}g_{\mu\beta}\right) + \frac{1}{4}Rg_{\mu\alpha\nu\beta} \\ &= \frac{1}{2}\left(R_{\mu\alpha\nu\beta} + {}^*R^*_{\mu\alpha\nu\beta}\right), \end{aligned}$$

$$G_{\mu\alpha\nu\beta} = -\frac{1}{12} R g_{\mu\alpha\nu\beta}, \qquad g_{\mu\alpha\nu\beta} \equiv g_{\mu\nu} g_{\alpha\beta} - g_{\mu\beta} g_{\nu\alpha}.$$

In the 8-dimensional bivector space (only 6 dimensions are essential), introducing the splitting tensors, we obtain

$$(R_{ab}) = \begin{pmatrix} X & Y \\ Y' & Z \end{pmatrix} = \begin{pmatrix} U & V \\ V & -U \end{pmatrix} + \begin{pmatrix} H & W \\ -W & H \end{pmatrix} + \begin{pmatrix} K & 0 \\ 0 & -K \end{pmatrix}.$$

Here

$$Y = (Y_{\mu\nu}), \qquad Y'(Y_{\nu\mu}) \text{ and so on}, \qquad Y_{\nu\mu} = R^*_{\mu\alpha\nu\beta} u^\alpha u^\beta,$$

$$H_{\mu\nu} = E_{\mu\alpha\nu\beta} u^\alpha u^\beta = \frac{1}{2}(X_{\mu\nu} + Z_{\mu\nu}) = -\frac{1}{2}(R_{\mu\nu} + \mathring{R} g_{\mu\nu}) + R_{(\mu} u_{\nu)} - \frac{1}{4} R (g_{\mu\nu} - u_\mu u_\nu),$$
$$\mathring{R} = R_{\mu\nu} u^\mu u^\nu, \qquad R_\mu = R_{\mu\nu} u^\nu,$$

$$V_{\mu\nu} = Y_{(\mu\nu)} \equiv \frac{1}{2}(Y_{\mu\nu} + Y_{\nu\mu}), \qquad W_{\mu\nu} = Y_{[\mu\nu]} = \frac{1}{2} \eta_{\mu\nu\rho\sigma} u^\rho R^\sigma_\tau u^\tau,$$

$$K_{\mu\nu} = G_{\mu\alpha\nu\beta} u^\alpha u^\beta = -\frac{1}{12} R (g_{\mu\nu} - u_\mu u_\nu).$$

When the tensors obtained from $R_{\mu\alpha\nu\beta}$ vanish, special Riemannian spaces arise: scalar-flat ($R = 0$), Einstein ($E_{\mu\alpha\nu\beta} = 0$, $R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu}$), conformally flat ($C_{\mu\alpha\nu\beta} = 0$), dually symmetric (${}^*R_{\mu\alpha\nu\beta} = R^*_{\mu\alpha\nu\beta}$, $W_{\mu\nu} = 0$), dually cosymmetric $R_{\mu\alpha\nu\beta} = -R^*_{\mu\alpha\nu\beta}$, $W_{\mu\nu} \neq 0$, and so on.

It is easily verified that the tensor $W_{\mu\nu}$ in dually nonsymmetric spaces vanishes if and only if the splitting vector is an eigenvector of the Ricci tensor.

The Ricci tensor, as is known (7), may belong to one of 3 types, depending on what its first eigenvector is: timelike (type a), isotropic (type b), or complex (type c). The canonical form of the tensor $E_{\mu\alpha\nu\beta}$ is obtained by a simple substitution into it of the canonical forms of the Ricci tensor, given, for example, in (7). Let us only note that for type a, H is diagonal, $W_{ik} = 0$; in type b there appear in H_{33} and H_{22} $\sigma/2$, $W_{32} = \sigma/2$ (for definiteness we set $\sigma = 1$); and in type c, $-W_{32} = W_{23} = -\lambda''/2$, where λ'' is the imaginary part of the first characteristic root of $R_{\mu\nu}$.

Reference systems canonical with respect to the tensor $E_{\mu\alpha\nu\beta}$ exist for all 3 of its types, whereas the corresponding real principal congruences exist only for the first two (the second canonical frame and the second principal congruence of the curvature tensor). The method for finding the congruences, as well as the boundaries of type change, is obvious.

Thus one may speak of the types Ia, Ib, Ic, Da, and so on, up to IIIc. Various special cases are possible in which the canonical axes of the tensors $C_{\mu\alpha\nu\beta}$ and $E_{\mu\alpha\nu\beta}$, and the characteristic roots, coincide.

The tensor $G_{\mu\alpha\nu\beta}$ has diagonal form in any orthoframe. The scalar R is equal to the sum of the eigenvalues of the Ricci tensor.

From a consideration analogous to the preceding one, we obtain that for $C_{\mu\alpha\nu\beta} = 0$, type a corresponds to dilatation, while in type b a transverse-transverse wave appears in addition.

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Note: Figure translations are in progress. See original paper for figures.