ON THE OSCILLATION OF THE SEQUENCE OF FEJÉR SUMS

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 517.512.2

MATHEMATICS

G. I. RYZHANKOVA

ON THE OSCILLATION OF THE SEQUENCE OF FEJÉR SUMS

(Presented by Academician S. L. Sobolev on 16 IX 1969)

Let $\sigma_n = \sigma_n(f;x)$ and $\sigma_m = \sigma_m(f;x)$ be the Fejér sums of order m and n $(m, n = 1, 2, ...)$ of a 2π-periodic function $f(x)$. In this paper we consider the problem of finding the least upper bound of the modulus of the difference $\sigma_m - \sigma_n$ on the classes $W^{(p)}$ $(p = 1, 2, 3, ...)$ of 2π-periodic functions that have an absolutely continuous derivative of order $(p-1)$ and a derivative of order p, bounded in modulus by one everywhere where this derivative exists, and also on the corresponding classes of conjugate functions. A similar problem, called the problem of oscillation, for the Abel-Poisson summation method was studied earlier in papers ([5,6]). Let

$$C_p = \frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu(p+1)}}{(2\nu+1)^p}, \qquad \widetilde{C}_p = \frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu p}}{(2\nu+1)^p}.$$

Theorem 1. For any two values m, n and $p \geq 2$, the equality

$$\sup_{f\in W^{(p)}} |\sigma_m - \sigma_n| = |\sigma_m(\varphi_p; x_p) - \sigma_n(\varphi_p; x_p)| \tag{1}$$

holds, where $x_p = [1 + (-1)^{p+1}]\pi/4$, and $\varphi_p(x)$ is a function from the class $W^{(p)}$ for which $\varphi_p^{(p)}(x) = \operatorname{sign}\cos x$.

Theorem 2. Uniformly with respect to m, n $(m \geq n)$, as $m, n \to \infty$, the following asymptotic equality holds:

$$\sup_{f\in W^{(p)}} |\sigma_m - \sigma_n| = \begin{cases} \dfrac{2(m-n)}{\pi m n}[\ln n + O(1)], & p = 1, \\ C_p \dfrac{m-n}{(m+1)(n+1)}[1 + O(n^{-p+1})], & p > 1. \end{cases} \tag{2}$$

Theorem 3. For any values $p \geq 2$, the equality

$$\sup_{f\in W^{(p)}} |\tilde{\sigma}_m(f;x) - \tilde{\sigma}_n(f;x)| = |\tilde{\sigma}_m(\varphi_p;x_{p+1}) - \tilde{\sigma}_n(\varphi_p;x_{p+1})| \tag{3}$$

holds.

Theorem 4. For any values $p \geq 2$, uniformly with respect to m and n ($m \geq n$), as $m, n \to \infty$, the following asymptotic equality holds:

$$\sup_{f\in W^{(p)}} |\tilde{\sigma}_m(f;x) - \tilde{\sigma}_n(f;x)| = \frac{\widetilde{C}_p(m-n)}{(m+1)(n+1)}[1 + O(n^{-p+1})]. \tag{4}$$

The proof of equality (1) is based on the fact that the difference $\sigma_m - \sigma_n$ admits a representation of the form

$$\frac{1}{\pi}\int_0^{2\pi} f^{(p)}(x+t)K(t)\,dt,$$

where

$$K(t) = \sum_{k=1}^{n} \frac{m-n}{(m+1)(n+1)k}\,\frac{\sin kt}{k} + \sum_{k=n+1}^{m} \left(\frac{1}{k^2} - \frac{1}{k(m+1)}\right)\frac{\sin kt}{k}. \tag{5}$$

Abel' s transformation makes it possible to show that expression (5) is a linear combination (with positive coefficients) of the functions

$$S_k(t) = \sum_{\nu=1}^{k} \frac{\sin \nu t}{\nu},$$

which are positive on $(0, \pi)$ (see (2), p. 106). Therefore the odd function $K(t)$ preserves its sign on $(0, \pi)$.

Equality (3) in the case $p \geq 3$ follows from (1), since it can be shown that the right-hand side of (1) is equal to

$$-\frac{4}{\pi(m+1)}\left|\frac{m-n}{n+1}\sum_{\nu=0}^{[(n-1)/2]} \frac{(-1)^{\nu(p+1)}}{(2\nu+1)^p} + \sum_{\nu=[(n-1)/2]+1}^{[(m-1)/2]} \frac{(-1)^{\nu(p+1)}(m-2\nu)}{(2\nu+1)^{p+1}}\right|.$$

In the case $p = 1, 2$, taking into account the equalities (see (2), p. 86)

$$\sum_{\nu=1}^{k} \sin \nu t = \frac{1-\cos kt}{2\,\mathrm{tg}\ t/2} + \frac{1}{2}\sin kt,$$

we obtain

$$\sigma_n - \sigma_m = \frac{1}{\pi} \int_0^{2\pi} f'(x+t) M(t)\, dt,$$

where

$$M(t) = \sum_{k=n+1}^{m} \frac{1 - \cos kt}{2k(k+1) \operatorname{tg} t/2} + \frac{1}{2} \sum_{k=n+1}^{m} \frac{\sin kt}{k(k+1)}.$$

The first sum in the preceding formula is an odd function, positive on $(0, \pi)$. Using this fact, it is not difficult to conclude that in the case $p = 1, 2$ the left-hand side of (1), up to the remainder term in equality (2), is equal to the right-hand side of (1). Thus, in this case as well, the proof of equality (2) is reduced to the computation of the right-hand side of (1).

The proof of Theorems 3 and 4 is carried out according to the same scheme. Estimates (2) and (4) are a generalization of the known results of S. M. Nikol' skii (4), which are obtained from these estimates by passing to the limit as $m \to \infty$.

Relations analogous to (1)–(4) hold also in the case when one considers the oscillations of the sequence of Fejér sums in the L-metric on classes with p-th derivative bounded in the L-metric.

Remark 1. Equality (1) for $p = 1, 2$ does not hold in general. However, for $p = 2$ it is valid for every n and all values of m, beginning with some value $m_0(n)$ depending on n. In the case $p = 1$, such an assertion is valid only for even values of n.

Remark 2. With the aid of the methods used to obtain the preceding results, one can show that on the functions $\varphi_p(x)$ the upper bound is also attained in problems on the oscillation of the sequence of de la Vallée-Poussin integrals

$$V_n = \frac{(2n)!!}{2\pi(2n-1)!!} \int_0^{2\pi} f(x+t) \cos^{2n}(t/2)\, dt$$

(on the classes $W^{(p)}$, $p \geq 1$); Jackson integrals

$$I_n = \frac{{}^3/_2 (n+1)^{-1}}{\pi\, [2(n+1)^2 + 1]} \int_0^{2\pi} f(x+t) \frac{\sin^4 (n+1)t/2}{\sin^4 t/2}\, dt,$$

Cesàro means $\sigma_n^{(\alpha)}(f; x)$ $(\alpha > 1)$, and Bernstein–Rogozinskii sums

$$B_n = \frac{1}{2} \left[S_n(f; x + \pi/2n) + S_n(f; x - \pi/2n) \right]$$

on the classes $W^{(p)}, p \geq 3$.

We shall give some asymptotic estimates related to this. Uniformly with respect to m and n $(m \geq n \geq 1)$, the following relations hold:

$$\sup_{f \in W^{(1)}} |V_m - V_n| = \frac{4}{\pi} \left\{ \frac{(2n)!!}{(2n+1)!!} - \frac{(2m)!!}{(2m+1)!!} \right\} + O\left(\frac{m-n}{mn}\right), \tag{6}$$

$$\sup_{f \in W^{(p)}\,(p \geq 3)} |I_m - I_n| = C(m,n,p)\,[1 + O(n^{-1} + n^{-p+2} \ln n)], \tag{7}$$

$$\sup_{f \in W^{(p)}\,(p \geq 3)} |B_m - B_n| = \frac{1}{12} \pi^2 C(m,n,p)\,[1 + O(n^{-2} + n^{-p+2})], \tag{8}$$

where $C(m,n,p) = \frac{3}{2}(m^2 - n^2) m^{-2} n^{-2} C_{p-1}$.

In the asymptotic estimate for the oscillations of the sequence of Cesàro means $\sigma_n^{(\alpha)}(f;x)$ $(\alpha > 2)$ on the classes $W^{(p)}$ $(p \geq 1)$, the leading term differs by the factor α from the leading term in estimate (2).

Equalities (6) and (8), with a somewhat different form of the remainder term in the case $m = \infty$, were obtained earlier in the works, respectively, ([3]) and ([1]).

In conclusion I express my deep gratitude to Prof. A. F. Timan for posing the problem and for his constant attention to the work.

Dnepropetrovsk Chemical-Technological
Institute

Received
4 VIII 1969

REFERENCES

[1] V. K. Dzyadyk, V. T. Gavrilyuk, A. I. Stepanets, *Dokl. AN USSR*, ser. A, No. 3 (1969).
[2] A. Zygmund, *Trigonometric Series*, 1, Moscow, 1965.
[3] I. P. Natanson, *DAN*, 54, No. 1 (1946).
[4] S. M. Nikol' skii, *Tr. Matem. inst. im. V. A. Steklova AN SSSR*, 15 (1945).
[5] A. F. Timan, V. N. Trofimov, *DAN*, 173, No. 4 (1967).
[6] V. N. Trofimov, A. S. Tsygankov, *Sibirsk. matem. zhurn.*, 10, No. 2 (1969).