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L. G. ZAMBACHIDZE

ON SOME PROPERTIES OF SPACES WITH BICOMPACT REMAINDERS OF FINITE ORDERS

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1. In this paper we study some properties of spaces of Iséki orders n, i.e. of the classes K_n $(n \geqslant -1)$, and of their mappings. The spaces are assumed to be completely regular and Hausdorff, and the mappings continuous.

As is known (^{1}), a space X belongs to the class K_n $(n \geqslant 0)$ in the sense of Iséki if and only if $R^{n/2}(X)$ is bicompact when n is even, and $R^{(n-1)/2}(X)$ is locally bicompact when n is odd. $R^i(X)$ is defined as follows: $R^0(X) = X$, $R^1(X)$ is the set of all points of non-local bicompactness of X, and $R^i(X) = R^1[R^{i-1}(X)]$ for $i > 1$.

In (^{1}) an "external" characteristic of the spaces of the classes K_n is also given: a space X belongs to the class K_n if and only if the remainder of order n (^{1}) is bicompact. It is clear that K_0 coincides with the class of bicompacts, and K_1 with the class of locally bicompact spaces. We shall further assume that the class K_{-1} consists of empty spaces.

In what follows another characteristic of the classes K_n will be needed.

Theorem 1. *A space X belongs to the class K_n $(n \geqslant 1)$ if and only if $R^1(X)$ belongs to the class K_{n-2}.*

It is shown in (^{1}) that a closed subset of a space from the class K_n also belongs to the class K_n. For open subsets the following holds.

Theorem 2. *Let X be a space of the class K_n, and let $G \subseteq X$ be an open subset of it. Then G belongs to the class K_{n+1} if n is even, and to the class K_n if n is odd.*

It is known that local bicompactness is preserved under open mappings. For spaces from the classes K_n the following holds.

Theorem 3. *The open image of a space of the class K_n belongs to the same class K_n.*

In (^{1}) the question of the topological product of spaces from the classes K_n was considered in the special case when one of the factors is either bicompact or locally bicompact. In the general case the following holds.

Theorem 4. *Let X belong to the class K_m, and Y to the class K_n; then $X \times Y$ belongs to the class K_{m+n} if either m or n is even, and to the class K_{m+n-1} if m and n are odd.*

The next theorem gives an internal characteristic of a space with a bicompact remainder of finite order, i.e. of some class K_n, and is an answer to a question posed by G. S. Chogoshvili.

Theorem 5. *A space X has a bicompact remainder of finite order if and only if it can be decomposed into the sum of a finite number of pairwise disjoint subsets, each of which is locally bicompact in the induced topology.*

For closed mappings of spaces from the classes the following theorems hold:

Theorem 6. *Let X be a hereditarily weakly paracompact space and let $f : X \to Y$ be a closed mapping. Then, if X belongs*

class K_n, then Y belongs to the class K_{2n} for even n and to the class K_{2n+1} for odd n.

Theorem 7. Let X be a hereditarily weakly paracompact space and $f : X \to Y$ a closed mapping such that $R(f)$ (see (1)) is bicompact. Then, if X belongs to the class K_n, then Y belongs to the class K_n for even n and to the class K_{n+1} for odd n.

The following theorem is a strengthening of a theorem of Lelek from (^{2}).

Theorem 8. Let X be a finally compact space of countable type and let $f : X \to Y$ be a mapping such that $f^{-1}(y)$ belongs to K_n for every $y \in Y$. Then

$$\dim X \leq \dim f(X) + \max\{\dim f, \operatorname{def} X\}.$$

Definition 1. A space X is called n-**peripheral** if in this space there exists a base of open sets whose boundaries belong to the class K_n.

The following theorem may be regarded as a generalization of the Freudenthal-Morita theorem (^{3}), obtained for $n = -1$.

Theorem 9. Let X be a space having a base of open sets $U = \{U_\alpha\}_{\alpha \in M}$ with the following properties: 1) the boundary $\operatorname{Fr} U_\alpha$ belongs to the class K_n for every $\alpha \in M$; 2) if $U_\alpha \in U$, then $X \setminus [U_\alpha] \in U$; 3) for every closed $A \subseteq U_\alpha \in U$ there exists $U_\beta \in U$ such that $A \subseteq [U_\beta] \subseteq U_\alpha$; 4) if $U_\alpha \in U$ and $U_\beta \in U$, then $U_\alpha \cup U_\beta \in U$. Then there exists a bicompact extension bX with an $(n-1)$-peripheral remainder $bX \setminus X$.

Definition 2. A base $U = \{U_\alpha\}_{\alpha \in M}$ of open sets of X is called **regular** if the following conditions are satisfied: 1) if $U_\alpha \in U$ and $U_\beta \in U$, then $U_\alpha \cup U_\beta \in U$; 2) if $U_\alpha \in U$, then $X \setminus [U_\alpha] \in U$; 3) if $U_\alpha \in U$ and $U_\beta \in U$, then $U_\alpha \setminus [U_\beta] \in U$.

If the Freudenthal-Morita theorem gives conditions under which there exists an extension with a zero-dimensional first remainder, then the following theorems give conditions for the existence of extensions with zero-dimensional remainders of the 2nd and 3rd orders. Zero-dimensionality is understood in the sense of ind.

Theorem 10. Let X be a paracompact 1-peripheral space. Then there exists an extension of order 2 with a zero-dimensional remainder.

Theorem 11. Let X be a paracompact space of countable type and let $U = \{U_\alpha\}_{\alpha \in M}$ be the collection of all open subsets of X whose boundaries are bicompactly situated in the sense of Chogoshvili (^{4}) and belong to the class K_2. If U is a regular base, then there exists an extension of order 3 with a zero-dimensional remainder.

The following theorem gives a condition under which spaces of the classes K_n are complete in the sense of Čech.

Theorem 12. Let X be a completely normal space of class K_n which is completely normally embedded (^{5}) in some bicompact extension. Then X is complete in the sense of Čech.

Corollary. A metrizable space of class K_n is an absolute G_δ.

Remark 1. For arbitrary completely normal spaces Theorem 12 is not true; there even exists a countable completely normal space of class K_2 which is not complete in the sense of Čech. However, the following holds.

Theorem 13. A countable space of class K_n is complete in the sense of Čech if and only if it is metrizable.

There exists a space of class K_n which is not a k-space. We give one sufficient condition under which a space of class K_n is a k-space.

Theorem 14. Let X be a space of class K_n such that for every bicompactum Φ from $R^1(X)$ there exists a bicompactum $F \supseteq \Phi$ of countable character in X. Then X is a k-space.

2. In the survey article (^{6}) A. V. Arhangel' skii posed the following problems:

Problem 1. Let X be a finally compact space and $f : X \to Y$ its closed mapping. Is the set

$$\Phi = \{y : Y \ f^{-1}(y) \text{ is not bicompact}\}$$

always at most countable?

Problem 2. Is every strict p-space perfectly mapped onto some space with a refining sequence of covers?

Below negative solutions of both of the indicated problems are given. Namely, in 2.1 an example will be constructed of a finally compact space of class K_2 in the sense of Isasardze, and of such a closed mapping $f : X \to Y$ of it that the set Φ is open, everywhere dense in Y, and has the cardinality of the continuum; in 2.2 an example will be constructed of a locally bicompact, weakly paracompact and σ-paracompact, nonnormal strict p-space which cannot be perfectly mapped onto any space with a refining sequence of covers. In constructing these examples one construction from (7) is used, applied there for another purpose.

2.1. Let $OXYZ$ be some rectangular coordinate system in R^3, and let K^3 be the unit cube:

$$K^3 = E\{(x, y, z) \in R^3 : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}.$$

Consider the opposite faces α and β of the cube K^3:

$$\alpha = E\{(x, y, z) \in K^3 : 0 \le x \le 1,\ y = 0,\ 0 \le z \le 1\}$$

and

$$\beta = E\{(x, y, z) \in K^3 : 0 \le x \le 1,\ y = 1,\ 0 \le z \le 1\}.$$

Let

$$\beta^* = E\{(x, y, z) \in \beta,\ z \text{ rational}\}.$$

We introduce a topology on the set $X = \alpha \cup \beta^*$ in the following way: if a point $M \in \beta^*$, then it is isolated in X; if, however, a point $M \in \alpha$, then its neighborhood is

$$OM = O'M \cup \{P(O'M) \setminus A\},$$

where $O'M$ is an open rectangular neighborhood of the point M in the usual topology of the face α; $P(O'M)$ is the projection onto β^*; and A is some countable subset of β^*. It can be shown that X is regular, finally compact, and belongs to the class K_2. Let

$$\Phi_1 = E\{(x, y, z) \in \alpha : 0 \le x \le 1,\ y = 0,\ z = 0\}$$

and

$$\Phi_2 = E\{(x, y, z) \in \beta^* : 0 \le x \le 1,\ y = 1,\ z = 0\}.$$

The set $Y = \Phi_1 \cup \Phi_2 \subseteq X$ is closed in X. Therefore $Y = \Phi_1 \cup \Phi_2$, in the induced topology, also has the listed properties of X. The mapping $f : X \to Y$ is defined as follows:

$$f(x, y, z) = (x, 1, 0), \quad \text{if } M(x, y, z) \in \beta^*,$$

and

$$f(x, y, z) = (x, 0, 0), \quad \text{if } M(x, y, z) \in \alpha.$$

It can be shown that the mapping f is closed, and the set Φ_2, consisting of the set of all points whose full inverse images are not bicompact, is open, everywhere dense in Y, and has the cardinality of the continuum.

2.2. Let $OXYZ$ be some rectangular coordinate system in R^3, and let K^3 be the unit cube:

$$K^3 = E\{(x, y, z) \in R^3 : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}.$$

Consider the sets α and β of the cube K^3, defined in the following way:

$$\alpha = E\{(x,y,z) \in K^3 : x \in C,\ y = 0,\ z \in C\}$$

and

$$\beta = E\{(x,y,z) \in K^3 : x \in C,\ y = 0,\ z \in C\},$$

where C denotes some fixed zero-dimensional compact subset of the interval $[0,1]$ of the cardinality of the continuum (for example, the Cantor perfect set). Define the set Φ as follows:

$$\Phi = E\{(x,y,z) \in \alpha : x \in C,\ y = 0,\ z = 0\}.$$

On the set

$$X = (\alpha \setminus \Phi) \cup \beta$$

we introduce a topology in the following way: represent β as the disjoint sum of the sets C_λ, i.e.

$$\beta \bigcup_{\lambda} C_\lambda, \quad C_\lambda \cap C_\gamma = \emptyset, \quad \text{for } \lambda \neq \gamma,$$

where each $C_\lambda = C$ has the topology induced from the interval $[0,1]$, parallel to the axis OZ. Let $x \in \beta$; then there exists a unique $C_{\lambda(x)}$ such that $x \in C_{\lambda(x)}$. As a neighborhood of the point x we declare an arbitrary open neighborhood of it in the topology of $C_{\lambda(x)}$. If the point $x \in \alpha \setminus \Phi$, then its neighborhood Ox in X is defined by the formula

$$Ox = O'x \cup \{P(O'x) \setminus B\}, \quad O'x = Vx \cap (\alpha \setminus \Phi),$$

where Vx is a rectangular neighborhood of the point x in the natural topology of the face α, having compact closure in α; $P(O'x)$ is the projection of $O'x$ onto β, and the set B is either empty or $\cdots$

$$B = \bigcup_{i=1}^{n} \Phi'_{\lambda_i},$$

where each $\Phi'_{\lambda_i} = T_i \cap C_{\lambda_i}$, and T_i is an open connected set on the interval $[0,1]$, containing C_{λ_i}.

It can be shown that the space X in this topology is locally bicompact, σ-paracompact, weakly paracompact, nonnormal, a strict p-space, and does not admit a perfect mapping onto a space with a refining sequence of covers.

Tbilisi State
University

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Note: Figure translations are in progress. See original paper for figures.