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Abstract

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MATHEMATICS

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ON A SINGULAR BOUNDARY VALUE PROBLEM FOR AN ORDINARY DIFFERENTIAL EQUATION OF THE n-TH ORDER

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In the present note, sufficient conditions are given for the solvability of the boundary value problem

$$u^{(n)} = f(t, u, u', \dots, u^{(n-1)}), \tag{1}$$

$$u^{(i-1)}(a) = u_{0i} \quad (i = 1, 2, \dots, n-1), \qquad u^{(m-1)}(b) = u_0, \tag{2}$$

where $1 \le m \le n-1$, $-\infty < a < b < +\infty$, $-\infty < u_{0i}, u_0 < +\infty$ $(i = 1, 2, \dots, n-1)$, and the function $f(t, x_1, \dots, x_n)$ is defined in the domain $a < t < b$, $-\infty < x_1, \dots, x_n < +\infty$, is measurable with respect to t, continuous with respect to $x_1, \dots, x_n$, and

$$f^*(t; \rho) = \sup\{|f(t, x_1, \dots, x_n)| : \ |x_k| \le \rho \ (k = 1, 2, \dots, n)\} \in$$

$$\in L(a + \delta, b - \delta)$$

for any $\rho \in (0, +\infty)$ and $\delta \in (0, b - a/2)$.

In contrast to the cases considered by other authors (see, for example, ([1],[2])), below it is assumed that the function $f(t, x_1, \dots, x_n)$, having singularities at $t = a$ and $t = b$, is in general not summable with respect to t on the interval $a < t < b$.

For what follows it is convenient to introduce the following

Definition. Let $\omega(t, x_1, \dots, x_k)$ be a nonnegative function defined in the domain $t_1 < t < t_2$, $-\infty < x_1, \dots, x_n < +\infty$. We shall say that it **belongs to**

the set $B_j^k(t_1,t_2)$, where $j \in \{1,2\}$, if for every $\rho \in (0,+\infty)$ there exists a nonnegative function $\varphi_\rho(t)$, continuous on the interval $t_1 < t < t_2$, such that $(t-t_j)^{k-1}\varphi_\rho(t) \in L(t_1,t_2)$ and $|v^{(k)}(t)| \le \varphi_\rho(t)$ for $\min\{t_0,t_j\} < t < \max\{t_0,t_j\}$, whatever the number $t_0 \in [t_1,t_2]$ and the function $v(t)$, absolutely continuous together with $v^{(i)}(t)$ $(i = 1,2,\dots,k)$ on the interval $t_1 \le t \le t_2$ and satisfying the conditions

$$|v^{(i-1)}(t)| \le \rho|t-t_j|^{1-i} \quad (i = 1,2,\dots,k),$$

$$v^{(k+1)}(t)\,\mathrm{sign}[(t_j-t)v^{(k)}(t)] \le \omega(t,v'(t),\dots,v^{(k)}(t)) \quad \text{for } t_1 < t < t_2,$$

$$|v^{(k)}(t_0)| < \rho.$$

By $D_r(t_1,t_2)$ below we denote the set

$$D_r(t_1,t_2) =$$

$$= \{t,x_1,\dots,x_n) :\ t_1 < t < t_2,\ |x_k| \le r\ (k = 1,\dots,m),$$

$$|x_k| \le r(b-t)^{m-k}\ (k = m+1,\dots,n-1),\ |x_n| < +\infty\}.$$

Theorem 1. *If*

$$f(t,x_1,\dots,x_{n-1},0)\,\mathrm{sign}\,x_{n-1} \ge 0 \quad \text{for } a < t < b,$$

$$-\infty < x_1,\dots,x_{n-2} < +\infty, \qquad r_0 \le |x_{n-1}| < +\infty,$$

$$f(t,x_1,\dots,x_n)\,\mathrm{sign}\,x_n \ge -\omega_1(t,x_n) \quad \text{for } (t,x_1,\dots,x_n) \in D_r(a,\beta), \qquad (3)$$

$$f(t,x_1,\dots,x_n)\,\mathrm{sign}\,x_n \le \omega_2(t,x_{m+1},\dots,x_n) \quad \text{for } (t,x_1,\dots,x_n) \in D(a,\beta),$$

where $a \le \alpha < \beta \le b$,

$$r_0 > 0, \quad r = (n-m)!(n-1)(1+b-a)^{n-2}\times$$
$$\times \max\{|u_{0i}|\ (i = 1,2,\dots,n-1),\ |u_0|,\ r_0\},$$

$$\omega_1(t,x_1) \in B_1^1(a,\beta), \qquad \omega_2(t,x_1,\dots,x_{n-m}) \in B_2^{n-m}(a,b), \tag{4}$$

then problem (1)–(2) is solvable.

With a special choice of the functions $\omega_1(t,x_1)$ and $\omega_2(t,x_1,\dots,x_{n-m})$, from Theorem 1 one can obtain a number of sufficient conditions for the solvability of problem (1)–(2). We give some of them.

Theorem 2. Let conditions (3) and (4) be satisfied,

$$f(t,x_1,\dots,x_n)\operatorname{sign} x_n \geq -h_1(t)(1+|x_n|)^{\lambda_1} \quad \text{for } (t,x_1,\dots,x_n) \in D_r(a,\beta), \tag{5}$$

$$f(t,x_1,\dots,x_n)\operatorname{sign} x_n \leq h_2(t)(1+|x_n|)^{\lambda_2} \quad \text{for } (t,x_1,\dots,x_n) \in D_r(a,\beta), \tag{6}$$

where $a \leq \alpha < \beta \leq b$, and λ_1 and $h_1(t)$ satisfy one of the following two conditions:

1) $\lambda_1 < 1, \quad h_1(t) \geq 0, \quad h_1(t) \in L(t_0,\beta)$ for every $t_0 \in (a,\beta)$, and

$$\left[\int_t^\beta h_1(\tau)\,d\tau\right]^{\frac{1}{1-\lambda_1}} \in L(a,\beta);$$

2) $1 \leq \lambda_1 \leq 2$ and

$$(1+|\ln(t-a)|)^{-1}h_1(t) \in L^{p_1}(a,\beta),$$

where

$$p_1 = \frac{1}{2-\lambda_1} \quad \text{if } \lambda_1 < 2, \qquad p_1 = +\infty \quad \text{if } \lambda_1 = 2,$$

and λ_2 and $h_2(t)$ satisfy one of the following two conditions:

1) $\lambda_2 < 1, \quad h_2(t) \geq 0, \quad h_2(t) \in L(a,t_0)$ for every $t_0 \in (a,b)$, and

$$(b-t)^{n-m-1}\left[\int_a^t h_2(\tau)\,d\tau\right]^{1/(1-\lambda_2)} \in L(a,b);$$

2) $1 \leq \lambda_2 \leq 2$ and

$$(b-t)^{(n-m-1)(1-\lambda_2)}(1+|\ln(b-t)|)^{-1}h_2(t) \in L^{p_2}(a,b),$$

where

$$p_2 = \frac{1}{2-\lambda_2} \quad \text{if } \lambda_2 < 2, \qquad p_2 = +\infty \quad \text{if } \lambda_2 = 2.$$

Then problem (1)–(2) is solvable.

Theorem 3. Let conditions (3), (4), and (5) be satisfied, where λ_1 and $h_1(t)$ satisfy the conditions of Theorem 2. Suppose further that

$$f(t, x_1, \dots, x_n) \operatorname{sign} x_n \le h_{20}(t) +$$

$$+ \sum_{k=1}^{n-m} h_{2k}(t)(1 + |x_{m+k}|)^{(n-m+1)/(k-1/kp_{2k})} \quad \text{for } (t, x_1, \dots, x_n) \in D_r(a, b),$$

where

$$a < \alpha < b, \qquad 1 \le p_{2k} < +\infty \quad (k = 1, 2, \dots, n-m),$$

$$(b-t)^{n-m} h_{20}(t) \in L(a, b), \qquad h_{2k}(t) \in L^{p_{2k}}(a, b) \quad (k = 1, 2, \dots, n-m).$$

Then problem (1)–(2) is solvable.

Theorem 4. Let conditions (3), (4), and (6) be satisfied, where $a < \alpha < b$, $\lambda_2 > 1$, the function $h_2(t)$ is positive, $h_2(t) \in L(a, b)$, and

$$\int_a^b (b-t)^{n-m-1} \left[\int_t^b h_2(\tau)\, d\tau \right]^{1-(1-\lambda_2)} dt = +\infty.$$

Suppose, further,

$$f(t, x_1, \dots, x_n) \operatorname{sign} x_n \ge -h_1(t)(1 + |x_n|)^{\lambda_1} \quad \text{for } (t, x, \dots, x_n) \in D_r(a, b),$$

where λ_1 and $h_1(t)$ satisfy the conditions of Theorem 2 for any $\beta \in (a, b)$. Then problem (1)–(2) is solvable.

Theorem 5. Suppose that conditions (3), (4), and (5) are satisfied, where $a < \beta < b$, $\lambda_1 > 1$, the function $h_1(t)$ is positive, $h_1(t) \in L(a, \beta)$, and

$$\int_a^\beta \left[\int_a^t h_1(\tau)\, d\tau \right]^{1/(1-\lambda_1)} dt = +\infty.$$

Suppose, further,

$$f(t, x_1, \dots, x_n) \operatorname{sign} x_n \le h_2(t)(1 + |x_n|)^{\lambda_2} \quad \text{for } (t, x_1, \dots, x_n) \in D_r(a, b),$$

where λ_2 and $h_2(t)$ satisfy the conditions of Theorem 2 for any $\alpha \in (a, b)$. Then problem (1)–(2) is solvable.

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Note: Figure translations are in progress. See original paper for figures.