THE KINETIC EQUATION IN THE PRESENCE OF NON-LAGRANGIAN FORCES

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Abstract

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THE KINETIC EQUATION IN THE PRESENCE OF NON-LAGRANGIAN FORCES

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The aim of the present work is to obtain an analogue of the Boltzmann equation for a system of N particles ($N \to \infty$), whose coordinates q_j and momenta p_j obey the quasicanonical equations:

$$\dot{q}_j = \partial H / \partial p_j; \qquad \dot{p}_j = -\partial H / \partial q_j + Q_j(p_j); \qquad j = 1, 2, \dots, N. \tag{1}$$

Here Q_j are non-Lagrangian forces, an example of which may be friction; H is the Hamiltonian function of the system obtained from the one under consideration if one sets $Q_j = 0$ ($j = 1, 2, \dots, N$).

As shown in works ([2,4]), for the s-particle correlation functions F_s the recurrence equations are valid:

$$\frac{\partial F_s}{\partial t} = \{H_s; F_s\} - \sum_{(1 \le j \le s)} \frac{\partial}{\partial p_j}(Q_j \cdot F_s) + \frac{1}{v} \int dx_{s+1} \left\{ \sum_{(1 \le j \le s)} \Phi_{j,s+1}; F_{s+1} \right\}. \tag{2}$$

Here the braces are Poisson brackets; $v = \lim(V/N)$ as $V, N \to \infty$; V is the volume occupied by the system in configuration space; Φ_{jk} is the interaction potential between the j-th and k-th particles. For brevity of notation we denote $x_j = (p_j, q_j)$, t is time, and H_s is the Hamiltonian function of a system of s particles.

To obtain the kinetic equation from system (2), we shall use an analogue of N. N. Bogolyubov' s reasoning ([1]). It can be shown that in the case under consideration all the assumptions sufficient for the evolution of the function $F_1(x_1, t)$ at the kinetic stage to be described by the equation

$$\partial F_1 / \partial t = A(x_1 | F_1), \tag{3}$$

where A is a quantity functionally dependent on F_1 and not depending explicitly on time, are valid. It is also assumed that for $s \geq 2$, $F_s = F_s(x_1, \dots, x_s|F_1)$. Put

$$A = \sum_{(0\leq j<\infty)} v^{-j} A_j(x_1|F_1); \qquad F_s = \sum_{(0\leq j<\infty)} v^{-j} F_s^{(j)}. \tag{4}$$

Substituting (4) into (2) for $s = 1$ and into (3), and then equating the coefficients of like powers of v, we obtain:

$$A_0 = -\left(\frac{p_1}{m}\frac{\partial}{\partial q_1} + \frac{\partial}{\partial p_1}Q_1\right)F_1; \qquad A_k = \int dx_2\{\Phi_{12}; F_2^{(k-1)}\} \quad (k > 0). \tag{5}$$

Introduce $D^{(r)}$, the operator of forming the derivative with respect to t followed by the substitution of $\partial F_1/\partial t$ by A_r. In this case the arguments of A_r and $\partial F_1/\partial t$ are assumed to coincide. Comparing the expression for $\partial F_s/\partial t$ obtained by substituting into the right-hand side of (2) the virial expansions (4) of the functions F_s and F_{s+1}, and the expression for $\partial F_s/\partial t$ written with the aid of the virial expansion (4) of F_s and the intro-

operator $D^{(r)}$, we obtain functional equations for $F_s^{(k)}$:

$$D^0 F_s^0 = \{H_s; F_s^0\} - \sum_{(1\leqslant j\leqslant s)} \frac{\partial}{\partial p_j}(Q_j(p_j)\cdot F_s^0);$$

$$D^{(1)} F_s^0 + D^0 F_s^{(1)} = \{H_s; F_s^{(1)}\} - \sum_{(1\leqslant j\leqslant s)} \frac{\partial}{\partial p_j}(Q_j \cdot F_s^{(1)}) +$$

$$+\int dx_{s+1}\left\{\sum_{(1\leqslant j\leqslant s)} \Phi_{j,s+1}; F_{s+1}^0\right\}; \tag{6}$$

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A detailed derivation of equations (6) is given in the preprint ([4]). The first of equations (6) are closed equations determining the set $\{F_s^0\}$. If $\{F_s^0\}$ is known, the second of equations (6) determine the set $\{F_s^{(1)}\}$, and so on. Thus, all equations (6) have the form

$$D^0 F_s^{(k)} = \{H_s; F_s^{(k)}\} - \sum_{(1\leqslant j\leqslant s)} \frac{\partial}{\partial p_j}(Q_{jF} s^{(k)}) - \Psi_s^{(k)}, \tag{6a}$$

where $\Psi_s^{(k)}$ are known functionals of F_1.

Introduce the k-particle evolution operator ([3])

$$S_k^*(t) = \exp\left[t \sum_{(1\leqslant j\leqslant k)} \left(\frac{\partial H_k}{\partial q_j}\frac{\partial}{\partial p_j} - \frac{\partial H_k}{\partial p_j}\frac{\partial}{\partial q_j} + \frac{\partial}{\partial p_j} Q_j\right)\right], \tag{7}$$

which coincides with the evolution operator of the Lagrangian system if in (7) one sets $Q_j = 0$. It has the properties ([3])

$$S_k^*(0) = 1, \quad S_k^*(t')S_k^*(t'') = S_k^*(t' + t''). \tag{8}$$

As is easy to see,

$$\frac{\partial}{\partial \tau} S_1^*(\tau)F_1(x_1, t) = \left(\frac{p_1}{m}\frac{\partial}{\partial q_1} + \frac{\partial}{\partial p_1} Q_1\right) S_k^*(\tau)F_1(x_1, t) = -A_0(x_1 \mid S_1^*(\tau)F_1). \tag{9}$$

From formula (9) and the definition of D^0 it follows that

$$D^0 F_s^{(k)}(x_1, \dots, x_s \mid S_1^*(\tau)F_1) = -\partial F_s^{(k)}/\partial \tau. \tag{10}$$

In view of the arbitrariness of F_1, replace in (6a) F_1 by $S_1^*(\tau)F_1$, after which, with the aid of (10), we arrive at the equations:

$$\frac{\partial F_s^{(k)}}{\partial \tau} = \{H_s; F_s^{(k)}\} + \sum_{(1\leqslant j\leqslant s)} \frac{\partial}{\partial p_j}(Q_j \cdot F_s^{(k)}) - \Psi_s^{(k)}(x_1, \dots, x_s \mid S_1^*(\tau)F_1);$$

$$F_s^{(k)} = F_s^{(k)}(x_1, \dots, x_s \mid S_1^*(\tau)F_1). \tag{11}$$

As boundary conditions we take the principle of weakening of correlations in the remote past:

$$\lim_{\tau\to\infty} S_m^*(-\tau)F_m(x_1, \dots, x_m \mid S_1^*(\tau)F_1) = \lim_{\tau\to\infty} S_m^*(-\tau) \prod_{(1\leqslant j\leqslant m)} S_1^*(\tau)F_1(x_j, t), \tag{12}$$

or, by virtue of (4):

$$\text{a)} \quad \lim_{\tau\to\infty} S_m^*(-\tau)F_m^0(x_1, x_2, \dots, x_m \mid S_1^*(\tau)F_1) =$$

$$= \lim_{\tau\to\infty} S_m^*(-\tau) \prod_{(1\leqslant j\leqslant m)} S_1^*(\tau) F_1(x_j, t);$$

$$\text{b)} \qquad \lim_{\tau\to\infty} S_m^*(-\tau) F_m^{(k)}(x_1, \dots, x_m \mid S_1^*(\tau)F_1) = 0 \quad (k > 0). \tag{13}$$

It can be shown that for a system with monotonically decreasing energy (for example, under the action of friction forces), the weakening of correlations between the dynamical states of particles in the remote past occurs more effectively than in a system whose energy is constant. When considering a system that increases its energy due to the action of forces Q_j, we shall assume an adiabatic inclusion of the forces Q_j at that stage of the system' s development when correlations were absent.

The solution of equation (11) for $k = 0$ has the form

$$F_m^0(x_1, \dots, x_m \mid S_1^*(\tau)F_1) = S_m^*(\tau) F_m^0(x_1, \dots, x_m \mid F_1). \tag{14}$$

Acting on both sides of (14) with the operator $S_m^*(-\tau)$, taking into account the properties (8), and then passing to the limit as $\tau \to \infty$, we obtain, on the basis of (13),

$$F_m^0(x_1, \dots, x_m \mid F_1) = T_m F_1(x_1, t) \dots F_1(x_m, t);$$

$$T_m = \lim_{\tau\to\infty} S_m^*(x_1, \dots, x_m, -\tau) \prod_{(1\leq j\leq m)} S_1^*(x_j, \tau). \tag{15}$$

Let us write the kinetic equation of the first approximation:

$$\frac{\partial F_1}{\partial t} = -\left[\frac{p_1}{m}\frac{\partial}{\partial q_1} + \frac{\partial}{\partial p_1} Q_1(p_1)\right] F_1 + \frac{1}{v} A_1(x_1, F_1);$$

$$A_1 = \int dx_2 \{\Phi_{12}; F_2^0(x_1, x_2 \mid F_1)\} = \int dx_2 \{\Phi_{12}; T_2 F_1(x_1, t) F_1(x_2, t)\}. \tag{16}$$

Denote by $X_1(x_1, x_2, t)$ and $X_2(x_1, x_2, t)$ the sets of coordinates and momenta that the first and second particles, respectively, would have at time t if, beginning from the "infinite past," they interacted only with each other (and were also subjected to the action of the forces $Q_{1,2}$).

Then

$$\{\Phi_{12}; T_2 F_1(x_1, t) F_1(x_2, t)\} = \{\Phi_{12}; F_1(X_1) F_1(X_2)\}. \tag{17}$$

Using the first of equations (6), relation (17), and also the definition of the operator D^0 and formula (5) for the operator A_0, after standard transformations analogous to those carried out in the monograph ([1]), we arrive at the following expression for A_1:

$$A_1 = I_B + \int dx_2 \left[Q(p_1) F_1(X_1) \frac{\partial}{\partial p_1} F_1(X_2) + Q(p_2) F_1(X_2) \frac{\partial}{\partial p_2} F_1(X_1) \right], \quad (18)$$

where I_B is the well-known expression for the Boltzmann collision integral ([1]). The integration in (18) is over the region of interaction of the particles in configuration space and over the layer inside the sphere $\vec{p} = p_{20}$ and $\vec{p} = p_{21}$ in momentum space. We have denoted by p_{20} and p_{21} the momenta of the second particle, respectively, at the beginning and at the end of the interaction.

Let us note that outside the sphere of interaction

$$\partial F_1(X_2)/\partial \vec{p}_1 = 0, \qquad \partial F_1(X_1)/\partial \vec{p}_2 = 0, \quad (19)$$

since in this case $X_1 = X_1(x_1)$ and $X_2 = X_2(x_2)$. Consequently, the integrand in formula (18) vanishes outside the region of interaction.

On the basis of (16) and (18), the final form of the kinetic equation of the first approximation is as follows:

$$\frac{\partial F_1}{\partial t} + \frac{\vec{p}_1}{m} \frac{\partial F_1}{\partial q_1} + \frac{\partial}{\partial p_1}(Q(p_1) F_1) = \frac{1}{v} \left[I_B + \int dx_2 \left[Q(p_1) F_1(X_1) \frac{\partial}{\partial p_1} F_1(X_2) + Q(p_2) F_1(X_2) \frac{\partial}{\partial p_2} F_1(X_1) \right] \right]. \quad (20)$$

If during the interaction the principal change in momenta occurs due to the action of intermolecular forces, and not the forces Q_j, then the last term on the right-hand side of (20) may be neglected in comparison with I_B.

In other words, if $Q_j = \alpha f(p_j)$, where α is a small dimensionless parameter, then in the equation of the first approximation one may neglect the term preceded by the product of two small parameters—(α/ν).

In conclusion, I express my gratitude to N. N. Bogolyubov for discussing the work.

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Note: Figure translations are in progress. See original paper for figures.