ON THE SOLUTION OF INTEGRAL EQUATIONS OF THE FORM

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Abstract

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ON THE SOLUTION OF INTEGRAL EQUATIONS OF THE FORM

$$\int_a^b K(x,s)\,dg(s) = u(x)$$

(Presented by Academician A. N. Tikhonov on 9 I 1970)

In the present paper we consider the question of solving the problem

$$A[x,g(s)] \equiv \int_a^b K(x,s)\,dg(s) = u(x);$$

$$g(a)=0; \qquad g(s)\in G; \qquad u(x)\in U; \qquad a\le s\le b,\ c\le x\le d, \tag{1}$$

where $u(x)$ is a given function, and $g(s)$ is the unknown function, belonging to certain functional spaces U and G, respectively. Let the kernel of equation (1), $K(x,s)$, be bounded and continuous on the rectangle $[ab]\times[cd]$, and suppose that for some $\bar{u}(x)\in U$ there exists, moreover uniquely, a function $\bar{g}(s)\in G$ such that $A[x,\bar{g}(s)]=\bar{u}(x)$ and $\bar{g}(a)=0$. It is evident that if G is the space of continuous functions with fixed left endpoint, then problem (1) is ill-posed ([1]).

1. Let an ill-posed problem (1) be given, and let it be known that the exact solution $\bar{g}(s)$ of the problem corresponding to $\bar{u}(x)$ is a monotone (for definiteness, nondecreasing), bounded function tending to 0 at the point a (to remove ambiguity we shall take the values of the function at all its interior discontinuity points to be equal to the limiting values, for example from the right). Suppose also that it is known that on the set of monotone functions with fixed left endpoint the solution of problem (1) for $u(x)=\bar{u}(x)$ is unique.

Introduce the set $G\uparrow$, whose elements are nondecreasing functions $g(s)$, with: a) $g(a)=0$; b) $\max_{a\le s\le b}|g(s)|<\infty$, and also the set of functions $G\uparrow(\bar{u},\delta)$, whose elements are functions from $G\uparrow$ satisfying the inequality

$$\|A[x,g(s)]-\bar{u}(x)\|_U \le \delta. \tag{2}$$

Theorem 1. There exists $\delta_0(A;\bar{g})$ such that for all $\delta < \delta_0$ the set $G \uparrow (\bar{u}, \delta)$ is contained inside the ball of radius 2 (the ball in the metric C). (Here and below, for definiteness, we put $\max_{a \le s \le b} |g(s)| = 1$.)

Proof. a) It is easy to see that the set $G \uparrow (\bar{u}, \delta)$ is convex for each fixed δ. From the convexity of the set $G \uparrow (\bar{u}, \delta)$, it is evident, follows the convexity of the numerical set of values of functions from $G \uparrow (\bar{u}, \delta)$ at the point b.

b) Suppose that there is a sequence δ_k such that the set $G \uparrow (\bar{u}, \delta_k)$ for each fixed δ_k is not contained in the ball of radius 2. From a) and b) it follows that for each fixed δ_k in $G \uparrow (\bar{u}, \delta_k)$ there exists a function $\bar{g}_{\delta_k}$ such that $\bar{g}_{\delta_k}(b) = 2$. But then from the set $\{\bar{g}_{\delta_k}\}$, by Helly' s theorem ([2]), one can choose a subsequence converging to some function $\bar{\bar{g}}(s)$, belonging to the same set and $\bar{\bar{g}}(b) = 2$. Passing to the limit in the inequality $\|A[x, \bar{g}_{\delta_k}(s)] - \bar{u}(x)\|_U \le \delta_k$, we obtain that $\|A[x, \bar{\bar{g}}] - \bar{u}(x)\|_U = 0$. The latter contradicts the uniqueness of the solution of problem (1).

Theorem 2. Let the sequence $\delta_k \to 0$. Then the sequence g_{δ_k}, where g_{δ_k} is an arbitrary element of $G \uparrow (\bar{u}, \delta_k)$, converges to $\bar{g}(s)$—the exact solution of (1)—everywhere on $[a, b]$.

From Theorems 1 and 2 it follows

Remark 1. For any $\varepsilon > 0$ there exists a $\delta_0(\varepsilon, A, \bar{g}(s))$ such that, for all $\delta < \delta_0$ and any function $\tilde{g}_\delta \in G \uparrow (\bar{u}, \delta)$, the inequality

$$\|\tilde{g}_\delta - \bar{g}\|_{L_p} < \varepsilon$$

holds.

Theorem 3. Let $\bar{g}(s) \in C_1$. Then for any $\varepsilon > 0$ there exists a $\delta_0(\varepsilon, A, \bar{g}(s))$ such that, for all $\tilde{g}_\delta(s) \in G \uparrow (\bar{u}, \delta)$,

$$\|\tilde{g}_\delta(s) - \bar{g}(s)\|_C \le \varepsilon$$

as soon as $\delta < \delta_0$.

2. Usually, in solving problem (1), what is known is not the exact value $\bar{u}(x)$, but certain $\tilde{u}_\delta(x)$ and $\delta > 0$ (obtained, say, from experiment) such that

$$\|\tilde{u}_\delta(x) - \bar{u}(x)\|_U \le \delta.$$

Introduce the set U_δ, whose elements are the functions $\tilde{u}_\delta(x)$ for which

$$\|\tilde{u}_\delta(x) - \bar{u}(x)\|_U \le \delta,$$

and the set $G \uparrow (\tilde{u}_\delta, \delta) \subset G \uparrow$, whose elements are functions $g_\delta(s) \in G \uparrow$ satisfying the inequality

$$\|A[x, g_\delta(s)] - \tilde{u}_\delta(x)\|_U \le \delta$$

for any fixed $\tilde{u}_\delta(x) \in U_\delta$.

Lemma 1. The set $G\uparrow(\tilde{u}_\delta, \delta) \subset G\uparrow(\bar{u}, 2\delta)$.

Indeed, from

$$\|A[x, g_\delta(s)] - \tilde{u}_\delta(x)\|_U \leq \delta$$

and

$$\|\tilde{u}_\delta(x) - \bar{u}(x)\|_U \leq \delta$$

it follows that

$$\|A[x, g_\delta(s)] - \bar{u}(x)\|_U \leq 2\delta.$$

It follows from Lemma 1 that all the theorems formulated above are also valid for the elements of the set $G\uparrow(\tilde{u}_\delta, \delta)$.

Remark 2. Since Helly's theorem is valid both for finite intervals and for infinite ones, Theorems 1 and 2 are valid, for example, in the case when $a = c = -\infty$, $b = d = +\infty$, provided the additional restriction is imposed on $K(x, s)$: for each fixed x,

$$K(x, \pm\infty) = 0.$$

3. For the practical determination of an approximate solution it is sufficient to find any function from $G\uparrow(\tilde{u}_\delta, \delta)$, i.e.

$$\|A[x, g_\delta(s)] - \tilde{u}_\delta(x)\|_U \leq \delta, \qquad g_\delta(s) \in G\uparrow, \tag{3}$$

which, after passage to a difference scheme, leads to the ordinary problem of convex programming ([3]). The only difference is that it is not necessary to find the minimum of the functional

$$\|A[x, g(s)] - \tilde{u}_\delta(x)\|_U,$$

but it is sufficient to construct a minimizing sequence until a function satisfying relation (3) is found. To construct the minimizing sequence one may apply methods entirely analogous to those which were used by the authors in solving Fredholm integral equations of the first kind on the set of monotone bounded functions ([4],[5]).

It is quite obvious that, owing to the convexity of the set of approximate solutions, the error estimate for an approximate solution of problem (1) can be carried out in the same way as the error estimate of the approximate solution described in ([4],[5]), while the practical methods for finding the error are analogous to the methods described in ([5]).

The method presented may find wide application in problems of mathematical statistics (for example, for finding the spectral function of noise from the correlation function), in problems of radio astronomy for determining the fluxes of radio emission from celestial radio sources, etc.

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Note: Figure translations are in progress. See original paper for figures.