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Abstract

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Hydromechanics

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ON THE STABILITY OF SOLITARY WAVES IN WEAKLY DISPERSIVE MEDIA

As is known ($^{1-3}$), a broad class of one-dimensional nonlinear waves in media with weak dispersion (for example, waves in shallow water, ion-acoustic and magnetoacoustic waves in plasma, etc.) is described by the Korteweg–de Vries equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{1}$$

This equation describes the evolution of a nonlinear wave in a coordinate system moving with the propagation velocity of long-wavelength disturbances c (for definiteness we shall call it the sound velocity). For waves in shallow water and for waves in plasma, u may be understood as a perturbation of velocity or pressure (in a simple wave they are unambiguously related to one another).

Equation (1) can be used with equal success both for media with negative dispersion, when the phase velocity of linear waves decreases with increasing wave number, and for media with positive dispersion, when the phase velocity increases with wave number. The only difference is that in the first case the coordinate x is measured in the direction of wave propagation, while in the second—in the opposite direction (^{2}).

The Korteweg–de Vries equation has by now been well studied ($^{1-4}$). In particular, it has been shown that in the evolution of arbitrary initial disturbances $u(x, 0)$ an important role is played by special solutions of equation (1) of the type of solitary waves, or solitons:

$$u = u_0(x, t) = af(\sqrt{a}(x - x_0)), \tag{2}$$

where a is the wave amplitude, $x = at$ is its phase, and the function $f(\xi)$, satisfying the following equations following from (1):

$$-f' + ff' + f''' = 0; \qquad -f + \frac{1}{2}f^2 + f'' = 0 \tag{3}$$

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(the prime denotes differentiation with respect to ξ), is equal to

$$f(\xi) = 3(\operatorname{ch}\xi/2)^{-2}. \tag{4}$$

A soliton is a one-dimensional nonlinear wave; under the assumption of one-dimensionality it is a completely stable formation. However, the question remains open whether the stability of a soliton is preserved under weak curvature, when its amplitude a and phase x_0 are slowly varying functions of the coordinate y measured across the direction of soliton propagation. The possibility of the development of an instability of the self-focusing type in periodic nonlinear waves indicates the need to investigate an analogous effect also in the case of a solitary wave.

When strict one-dimensionality is violated, equation (1) is perturbed, and if the dependence on the coordinate y is slow, this perturbation may be taken into account by adding to equation (1) a small term $\partial\varphi/\partial y$:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = \frac{\partial \varphi}{\partial y}. \tag{5}$$

The form of the perturbation is easy to establish by considering the limiting case of a plane wave of small amplitude $u = \exp(-i\omega t + i\mathbf{kr})$ with small wavelength along x, $k_x \ll 1$, when the second and third terms in (5) may be neglected. For such a wave, for negative dispersion, the frequency of oscillations in the coordinate system used by us, moving along the x-axis with velocity c, is

$$\omega = kc - k_{xc} = c\left(\sqrt{k_x^2 + k_y^2} - k_x\right) \simeq \frac{c}{2}\frac{k_y^2}{k_x},$$

whereas for a wave with positive dispersion, since in this case x is measured against the wave,

$$\omega = -kc + k_{xc} \simeq -\frac{c}{2}\frac{k_y^2}{k_x}.$$

It follows that

$$\frac{\partial \varphi}{\partial x} = \mp\frac{c}{2}\frac{\partial u}{\partial y}, \tag{6}$$

where the upper sign corresponds to negative dispersion, and the lower to positive dispersion.

We shall henceforth restrict ourselves to the linear approximation, assuming $\partial x_0/\partial y \ll 1$ (although the amplitude of oscillations of the phase x_0 may still be considerably larger than the soliton width $\sim 1/\sqrt{a}$). But even in the linear approximation the system of equations (5), (6) is rather complicated, and therefore we pass to the limiting case of very large wavelengths along y, when $\partial x_0/\partial y \ll \sqrt{a}/c < 1$. In this case φ is small, and equations (5), (6) may be solved by the Krylov–Bogolyubov method, i.e., by introducing slowly varying

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variables. For this purpose, in equation (5), instead of x we introduce the variable $\xi = \sqrt{a}(x - x_0)$ and write it in the form

$$a^{3/2}\frac{\partial}{\partial \xi}\left(-u + \frac{u^2}{2a} + \frac{\partial^2 u}{\partial \xi^2}\right) = \sqrt{a}\,\frac{\partial u}{\partial \xi}\left(\frac{\partial x_0}{\partial t} - a\right) - \frac{\partial u}{\partial t} - \frac{1}{2}\frac{\partial a}{\partial t}\,\xi\frac{\partial u}{\partial \xi} + \frac{\partial \varphi}{\partial y}, \quad (7)$$

where the right-hand side may be regarded as small.

In the zeroth approximation we set the right-hand side equal to zero and obtain $u = u_0 = af(\xi)$. In the next approximation we put $u = u_0 + u_1$ and linearize the left-hand side of equation (7), while substituting u_0 into the right-hand side:

$$a^{3/2}\frac{\partial}{\partial \xi}\left(-u_1 + fu_1 + \frac{\partial^2 u_1}{\partial \xi^2}\right) = a^{3/2}f'\left(\frac{\partial x_0}{\partial t} - a\right) - \frac{\partial a}{\partial t}\,z + \frac{\partial \varphi}{\partial y}, \qquad (8)$$

where $f' = df/d\xi$, $z = f + \frac{1}{2}\xi f'$, and φ is to be found from (6) with substitution of $u = u_0$ in the right-hand side.

We shall regard $\partial/\partial t$ and $\partial/\partial y$ as small of order ε. As we shall see below, the oscillations of the phase $x_0(y)$ are considerably larger than the oscillations of the amplitude a, so that the variable part of the amplitude $\tilde{a}$ should be regarded as small of order $\varepsilon\tilde{x}_0$. To accuracy up to terms of order ε, in the right-hand side of equation (8) one should retain only the first term. In this case, as is not difficult to verify with the aid of (3), from equation (8) we would find, as the solution,

$$u_1 = \left(\frac{\partial x_0}{\partial t} - a\right)z,$$

where $z = f + \frac{1}{2}\xi f'$. But the addition proportional to z to the basic solution u_0 corresponds, as is easy to see, to a small variation of the soliton amplitude a. Without loss of generality we may require that this addition vanish; this will correspond simply to the correct choice of the value of the amplitude. (In exactly the same way one may require that the part of the perturbation $u_1 = \mathrm{const}\cdot f'$, which is a solution of the homogeneous equation (8), vanish; this corresponds to the correct choice of the phase x_0.) Thus, to accuracy up to small quantities of second order in ε, we have

$$\partial x_0/\partial t = a. \qquad (9)$$

In the next approximation in ε, it suffices for us to take into account in φ_1 only terms of first order of smallness in φ_1, so that in the linear approximation

$$\varphi_1 = \pm\frac{ca}{2}\frac{\partial x_0}{\partial y}f.$$

Consequently, for the correction of the second order of smallness u_2 we have

$$\frac{\partial u_2}{\partial t} + a^{3/2}\frac{\partial}{\partial \xi}\left(-u_2 + fu_2 + \frac{\partial^2 u_2}{\partial \xi^2}\right) = -\frac{\partial a}{\partial t}z \pm \frac{ca}{2}\frac{\partial^2 x_0}{\partial y^2}f. \qquad (10)$$

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Multiplying equation (10) by f and integrating with respect to ξ, we find that, as is not difficult to verify with account of (3), the integral of the second term on the left-hand side of (10), after integration by parts, vanishes; and from the smallness condition $\partial u_2/\partial t \sim \varepsilon^3$ it follows, to within terms of the third order of smallness, that

$$\frac{\partial a}{\partial t} = \pm\frac{2ca}{3}\frac{\partial^2 x_0}{\partial y^2}, \tag{11}$$

where the factor 2/3 arose after averaging over ξ (taking into account $\langle zf\rangle = 3/4\langle f^2\rangle$). From (11) it is clear that the oscillations of the amplitude are indeed significantly smaller than the oscillations of the phase $\tilde{x}_0$.

The function u_2 can be found approximately if one sets $u_2 = A(1-z)$ and regards A as a slowly varying function of ξ, i.e., neglects its higher derivatives with respect to ξ. In this approximation, with account of (11), equation (10) takes the form

$$(1-z)\frac{\partial A}{\partial t} + a^{3/2}\frac{\partial A}{\partial \xi} = -\frac{1}{4}\frac{\partial a}{\partial t}(f + \xi f'). \tag{12}$$

Outside the soliton, this equation describes a wave running opposite to ξ:

$$\partial A/\partial t + a^{3/2}\partial A/\partial \xi = 0, \tag{13}$$

so that $A = F(t - \xi a^{-3/2})$, where F is an arbitrary function. In the region of the soliton in equation (12), one may neglect the derivative of A with respect to time; and if in the stream incident on the soliton the perturbation is absent, then A is obtained from (12) by integration with respect to ξ with the boundary condition $A = 0$ at $\xi = \infty$. In particular, on the left outside the soliton we obtain

$$A(t) = F(t) \simeq -\frac{1}{4}\frac{\partial a}{\partial t} \sim \varepsilon^2.$$

Thus, when the soliton oscillates, a long-wave perturbation of the running-wave type, with amplitude proportional to $\partial a/\partial t$, propagates backward away from it.

From equations (9), (11) we find the equation for the oscillations of the phase of the soliton x_0:

$$\frac{\partial^2 x_0}{\partial t^2} = \pm\frac{2ca}{3}\frac{\partial^2 x_0}{\partial y^2}, \tag{14}$$

where the amplitude a, in the approximation considered by us, may be regarded as constant.

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It is seen from equation (14) that in a medium with positive dispersion (minus sign) the soliton is unstable with respect to curvature, and its small perturbations will grow with time. The increment of growth of such perturbations is proportional to the square root of the amplitude, i.e., it is rather large.

In the case of negative dispersion (plus sign), equation (14) leads to harmonic oscillations, and in order to resolve the question of stability or instability of the soliton one must take into account terms of order ε^3. For this purpose, in equation (10) one should retain the addition of the second order of smallness φ_2, satisfying the relation

$$\frac{\partial\varphi_2}{\partial\xi} = -\frac{c}{2\sqrt{a}}\frac{\partial a}{\partial y}z. \tag{15}$$

Taking into account that in the flow incident on the soliton the perturbation is absent, we have $\varphi = 0$ for $\xi = +\infty$, and from (15) we easily find φ_2. Multiplying again (10) by f and integrating with respect to ξ, retaining the small terms $\partial u_2/\partial t$ and $\partial\varphi_2/\partial y$, one can verify that small additions $\sim \varepsilon^3$ lead to damping of the soliton oscillations.

Thus, we have shown that in the case of negative dispersion (for example, for waves on shallow water) the "bending" of a soliton leads to elastic oscillations with weak damping. In the case of positive dispersion, the soliton is unstable with respect to two-dimensional perturbations of the type of its bending, and it is unlikely that it can exist for a long time.

If, however, in equation (1) the nonlinear term has a negative sign, then the situation is the reverse: solitons are stable in the case of positive dispersion, and unstable in the case of negative dispersion.

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