ON THE CARDINALITY OF SOME CLASSES OF GRAPHS

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Abstract

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ON THE CARDINALITY OF SOME CLASSES OF GRAPHS

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1°. In the present note asymptotic expressions are established for the cardinality of some classes of graphs with a large number of edges and vertices. Analogous expressions are established for the cardinality of similar classes of directed graphs. Before introducing these classes and formulating the results, we shall give explanations of the concepts used. We adhere to the terminology of [1], with the following exception: instead of general graph, multigraph, and unilaterally connected, we use respectively the terms multigraph, multigraph without loops, and connected. By a graph we mean a multigraph without multiple edges, but, generally speaking, with loops. By a directed multigraph we mean an object obtained from a multigraph by orienting all edges. A directed graph is such a directed multigraph in which each ordered pair of vertices (u, v) (possibly $u = v$) is connected by at most one arc leaving u and entering v. An initially connected multigraph is such a directed multigraph D with a distinguished (initial) vertex from which all the other vertices of D are reachable.

2°. We now introduce the classes under study. First, we consider 8 classes $\mathfrak{G}_{1,\sigma_2,\sigma_3}(n,k)$ of multigraphs with n numbered vertices and k edges, defined as follows. Let $\mathfrak{G}_1(n,k)$ be the set of all n-vertex multigraphs with k edges, in which the vertices are numbered by the numbers $1, 2, \ldots, n$;

$$\mathfrak{G}^1_{1,1}(n,k),\ \mathfrak{G}^2_{1,1}(n,k),\ \mathfrak{G}^2_{1,2}(n,k),\ \mathfrak{G}^2_{1,3}(n,k)$$

are subsets of this set consisting respectively of connected multigraphs, graphs without loops, graphs with loops, and multigraphs without loops;

$$\mathfrak{G}^1_{1,0}(n,k) = \mathfrak{G}^2_{1,0}(n,k) = \mathfrak{G}_1(n,k).$$

Then

$$\mathfrak{G}_{1,\sigma_2,\sigma_3}(n,k) = \mathfrak{G}^1_{1,\sigma_2}(n,k) \cap \mathfrak{G}^2_{1,\sigma_3}(n,k),$$

where $\sigma_2 = 0, 1,\ \sigma_3 = 0, 1, 2, 3$.

For example, $\mathfrak{G}_{1,0,0}(n,k) = \mathfrak{G}_1(n,k)$, while $\mathfrak{G}_{1,1,2}(n,k)$ is the set of all connected graphs with loops from $\mathfrak{G}_1(n,k)$.

Second, we consider 16 classes $\mathfrak{G}_{2,\sigma_2,\sigma_3}(n,k)$ of directed multigraphs with n numbered vertices and k arcs, defined as follows. Let $\mathfrak{G}_2(n,k)$ be the set of all n-vertex directed multigraphs with k arcs, in which the vertices are numbered by the numbers

$$1, 2, \dots, n;\ \mathfrak{G}^1_{2,1}(n,k),\ \mathfrak{G}^1_{2,2}(n,k),\ \mathfrak{G}^1_{2,3}(n,k),\ \mathfrak{G}^2_{2,1}(n,k),\ \mathfrak{G}^2_{2,2}(n,k),\ \mathfrak{G}^2_{2,3}(n,k)$$

are subsets of this set consisting respectively of weakly connected, strongly connected directed multigraphs, directed graphs without loops, directed graphs with loops, and multigraphs without loops; $\mathfrak{G}^1_{2,0}(n,k) = \mathfrak{G}^2_{2,0}(n,k) = \mathfrak{G}_2(n,k)$.

Then

$$\mathfrak{G}_{2,\sigma_2,\sigma_3}(n,k) = \mathfrak{G}^1_{2,\sigma_2}(n,k) \cap \mathfrak{G}^2_{2,\sigma_3}(n,k),$$

where $\sigma_2, \sigma_3 = 0, 1, 2, 3$.

Third, we consider the 4 classes $\mathfrak{G}_{3,0,0}(n,k)$, $\mathfrak{G}_{3,0,1}(n,k)$, $\mathfrak{G}_{3,0,2}(n,k)$, and $\mathfrak{G}_{3,0,3}(n,k)$ of n-vertex initially connected multigraphs, in which k arcs are contained and the vertices are numbered by the numbers $1, 2, \dots, n$. These classes consist, respectively, of multigraphs with loops, graphs without loops, graphs with loops, and multigraphs without loops.

Fourth, we shall consider 28 classes $\mathfrak{H}_{\sigma_1,\sigma_2,\sigma_3}(n,k)$, defined as follows. $\mathfrak{H}_{\sigma_1,\sigma_2,\sigma_3}(n,k)$ consists of the set of isomorphism classes associated with multigraphs from $\mathfrak{G}_{\sigma_1,\sigma_2,\sigma_3}(n,k)$. The number of elements in $\mathfrak{G}_{\sigma_1,\sigma_2,\sigma_3}(n,k)$ and $\mathfrak{H}_{\sigma_1,\sigma_2,\sigma_3}(n,k)$ will be denoted, respectively, by $G_{\sigma_1,\sigma_2,\sigma_3}(n,k)$ and $H_{\sigma_1,\sigma_2,\sigma_3}(n,k)$.

3°. It is not difficult to verify that, for any n and any admissible k, the following relations hold:

$$\begin{aligned}
G_{1,0,0}(n,k) &= C^k_{(n+1)n/2+k-1}, & G_{1,0,1}(n,k) &= C^k_{n(n-1)/2},\\
G_{1,0,2}(n,k) &= C^k_{(n+1)n/2}, & G_{1,0,3}(n,k) &= C^k_{n(n-1)/2+k-1},\\
G_{2,0,0}(n,k) &= C^k_{n^2+k-1}, & G_{2,0,1}(n,k) &= C^k_{n(n-1)},\\
G_{2,0,2}(n,k) &= C^k_{n^2}, & G_{2,0,3}(n,k) &= C^k_{n(n-1)+k-1}.
\end{aligned}$$

We now formulate the results. By $\lambda(n)$, everywhere unless otherwise specified, we mean an arbitrary function satisfying the inequality $\lambda(n) \geq -\ln\ln n + 1$.

Theorem 1. *For any admissible* $k = [\frac{1}{2}n(\ln n + \lambda(n))]$ *and for any pair* (σ_1, σ_3)*, where* $\sigma_1 = 1, 2,$ *and* $\sigma_3 = 0, 1, 2, 3,$

$$G_{\sigma_1,1,\sigma_3}(n,k) = G_{\sigma_1,0,\sigma_3}(n,k)\exp\{-\exp(-\lambda(n))\}(1 + o(1)).$$

A special case of this theorem was established by P. Erdős and A. Rényi ([2]).

Corollary 1. *If* $\lambda(n) \to +\infty$ *as* $n \to +\infty$*, then for any admissible* $k = [\frac{1}{2}n(\ln n + \lambda(n))]$ *and any pair* (σ_1, σ_3)*,* $\sigma_1 = 1, 2,$ $\sigma_3 = 0, 1, 2, 3,$

$$G_{\sigma_1,1,\sigma_3}(n,k) \sim G_{\sigma_1,0,\sigma_3}(n,k).$$

Theorem 2. *For any admissible* $k = [n(\ln n + \lambda(n))]$ *and any* $\sigma_3 = 0, 1, 2, 3,$

$$G_{2,2,\sigma_3}(n,k) = G_{2,0,\sigma_3}(n,k)\exp\{-2\exp(-\lambda(n))\}\times$$

$$\times\{1 + 2\exp(-\lambda(n)) + \exp(-2\lambda(n))\}(1 + o(1)).$$

Theorem 3. *For any admissible* $k = [n(\ln n + \lambda(n))]$ *and any* $\sigma_3 = 0, 1, 2, 3,$

$$G_{2,3,\sigma_3}(n,k) = G_{2,0,\sigma_3}(n,k)\exp\{-2\exp(-\lambda(n))\}(1 + o(1)).$$

A special case of Theorem 3 was established by I. Palásti ([3]).

From Theorems 2 and 3 we obtain

Corollary 2. *If* $\lambda(n) \to +\infty$ *as* $n \to +\infty$*, then for any admissible* $k = [n(\ln n + \lambda(n))]$ *and any pair* (σ_2, σ_3)*,* $\sigma_2 = 2, 3,$ *and* $\sigma_3 = 0, 1, 2, 3,$

$$G_{2,\sigma_2,\sigma_3}(n,k) \sim G_{2,0,\sigma_3}(n,k).$$

Theorem 4. *For any admissible* $k = [n(\ln n + \lambda(n))]$ *and any* $\sigma_3 = 0, 1, 2, 3,$

$$G_{3,0,\sigma_3}(n,k) = G_{2,0,\sigma_3}(n,k)n\exp\{-\exp(-\lambda(n))\}(1 + o(1)).$$

Corollary 3. *If* $\lambda(n) \to +\infty$ *as* $n \to +\infty$*, then for any admissible* $k = [n(\ln n + \lambda(n))]$ *and any* $\sigma_3 = 0, 1, 2, 3,$

$$G_{3,0,\sigma_3}(n,k) \sim nG_{2,0,\sigma_3}(n,k).$$

4°. In this item we shall formulate results concerning non-isomorphic multigraphs.

Theorem 5. Let c be an arbitrary positive constant, and let $\lambda(n)$ satisfy the relation $c \le \lambda(n) \le n/2$. Then:

1) for $k = [\frac{1}{2}n(\ln n + \lambda(n))]$ and $\sigma_3 = 1, 2$,

$$H_{1,0,\sigma_3}(n,k) = G_{1,0,\sigma_3}(n,k)/(n! \exp\{\exp(-\lambda(n))\})\times$$
$$\times(1 - \exp(-\lambda(n)))(1 + o(1));$$

2) for $k = C_n^2 - r$, where $r = [\frac{1}{2}n(\ln n + \lambda(n))]$,

$$H_{1,0,1}(n,k) = G_{1,0,1}(n,r)/(n! \exp\{\exp(-\lambda(n))\})\times$$
$$\times(1 - \exp(-\lambda(n)))(1 + o(1));$$

3) for $k = C_{n+1}^2 - r$, where $r = [\frac{1}{2}n(\ln n + \lambda(n))]$,

$$H_{1,0,2}(n,k) = G_{1,0,2}(n,r)/(n! \exp\{\exp(-\lambda(n))\})\times$$
$$\times(1 - \exp(-\lambda(n)))(1 + o(1)).$$

Corollary 4. If $\lambda(n) \to +\infty$ as $n \to +\infty$ and $\lambda(n) < n/2$, and k satisfies the relation

$$\frac{1}{2}n(\ln n + \lambda(n)) \le k \le C_n^2 - \frac{1}{2}n(\ln n + \lambda(n)),$$

then

$$H_{1,0,1}(n,k) \sim G_{1,0,1}(n,k)/n!$$

Various special cases of this corollary were established by several authors ([4–9]).

Corollary 5. If $\lambda(n) \to +\infty$ as $n \to +\infty$ and $\lambda(n) < n/2$, and k satisfies the relation

$$\frac{1}{2}n(\ln n + \lambda(n)) \le k \le C_{n+1}^2 - \frac{1}{2}n(\ln n + \lambda(n)),$$

then

$$H_{1,0,2}(n,k) \sim G_{1,0,2}(n,k)/n!$$

Theorem 6. Let c be an arbitrary positive constant, and let $\lambda(n)$ satisfy the relation $c \le \lambda(n) \le n$. Then:

1) for $k = [\frac{1}{2}n(\ln n + \lambda(n))]$ and $\sigma_3 = 1, 2$,

$$H_{2,0,\sigma_3}(n,k) = G_{2,0,\sigma_3}(n,k)/(n! \exp\{\exp(-\lambda(n))\})\times$$
$$\times(1 - \exp(-\lambda(n)))(1 + o(1)).$$

2) for $k = n(n-1) - r$, where $r = [\frac{1}{2}n(\ln n + \lambda(n))]$,

$$H_{2,0,1}(n,k) = G_{2,0,1}(n,r)/(n! \exp\{\exp(-\lambda(n))\})\times$$
$$\times(1 - \exp(-\lambda(n)))(1 + o(1)).$$

3) for $k = n^2 - r$, where $r = [\frac{1}{2}n(\ln n + \lambda(n))]$,

$$H_{2,0,2}(n,k) = G_{2,0,2}(n,r)/(n! \exp\{\exp(-\lambda(n))\})\times$$
$$\times(1 - \exp(-\lambda(n)))(1 + o(1)).$$

Corollary 6. If $\lambda(n) \to +\infty$ as $n \to +\infty$ and $\lambda(n) < n$, and k satisfies the relation

$$\frac{1}{2}n(\ln n + \lambda(n)) \leq k \leq n^2 - \frac{1}{2}n(\ln n + \lambda(n)),$$

then

$$H_{2,0,2}(n,k) \sim G_{2,0,2}(n,k)/n!$$

A special case of Corollary 6 was proved by Obershelp ([7]).

Theorem 7. Let c be an arbitrary positive constant, and let $\lambda(n) \geq c$. Then, for $k = [\frac{1}{2}n(\ln n + \lambda(n))]$ and for any choice of (σ_1, σ_3) such that $\sigma_1 = 1, 2$, and $\sigma_3 = 0, 3$,

$$H_{\sigma_1,0,\sigma_3}(n,k) = G_{\sigma_1,0,\sigma_3}(n,k)/(n! \exp\{\exp(-\lambda(n))\})\times$$
$$\times(1 - \exp(-\lambda(n)))(1 + o(1)).$$

Corollary 7. If $\lambda(n) \to +\infty$ as $n \to +\infty$, $k = [\frac{1}{2}n(\ln n + \lambda(n))]$, then for any choice of (σ_1, σ_3) such that $\sigma_1 = 1, 2$, and $\sigma_3 = 0, 3$,

$$H_{\sigma_1,0,\sigma_3}(n,k) \sim G_{\sigma_1,0,\sigma_3}(n,k)/n!$$

Let $\varphi(n) = \frac{1}{2}n(\ln n - \ln\ln n + 1)$.

Theorem 8. 1) If k satisfies the relation $\varphi(n) \leq k \leq C_n^2 - \varphi(n)$, then

$$H_{1,1,1}(n,k) \sim G_{1,1,1}(n,k)/n!$$

2) If k satisfies the relation $\varphi(n) \leq k \leq C_{n+1}^2 - \varphi(n)$, then

$$H_{1,1,2}(n,k) \sim G_{1,1,2}(n,k)/n!$$

3) If k satisfies the relation $\varphi(n) \leq k \leq n(n-1) - \varphi(n)$, then

$$H_{2,1,1}(n,k) \sim G_{2,1,1}(n,k)/n!$$

4) If k satisfies the relation $\varphi(n) \leq k \leq n^2 - \varphi(n)$, then

$$H_{2,1,2}(n,k) \sim G_{2,1,2}(n,k)/n!$$

Theorem 9. 1) If k satisfies the relation $2\varphi(n) \le k \le n(n-1) - 2\varphi(n)$, then

$$H_{3,0,1}(n,k) \sim G_{3,0,1}(n,k)/n!,$$

and for $\sigma_2 = 2, 3$

$$H_{2,\sigma_2,1}(n,k) \sim G_{2,\sigma_2,1}(n,k)/n!$$

2) If k satisfies the relation $2\varphi(n) \le k \le n^2 - 2\varphi(n)$, then

$$H_{3,0,2}(n,k) \sim G_{3,0,2}(n,k)/n!,$$

and for $\sigma_2 = 2, 3$

$$H_{2,\sigma_2,2}(n,k) \sim G_{2,\sigma_2,2}(n,k)/n!$$

3) If $k \ge 2\varphi(n)$, then for $\sigma_3 = 0, 3$

$$H_{3,0,\sigma_3}(n,k) \sim G_{3,0,\sigma_3}(n,k)/n!,$$

and for pairs (σ_2, σ_3) such that $\sigma_2 = 2, 3,\ \sigma_3 = 0, 3$,

$$H_{2,\sigma_2,\sigma_3}(n,k) \sim G_{2,\sigma_2,\sigma_3}(n,k)/n!$$

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Note: Figure translations are in progress. See original paper for figures.