ORTHOGONAL SYSTEMS OF ELEMENTS IN SMOOTH BANACH SPACES

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Abstract

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MATHEMATICS

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ORTHOGONAL SYSTEMS OF ELEMENTS IN SMOOTH BANACH SPACES

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1. It is known that if a system of elements (e_k) is orthonormal in a Hilbert space, then it forms an unconditional basis in the closure $[e_k]$ of its linear span. The notion of orthogonality can also be introduced in an arbitrary Banach space ($^{1-3}$). Slightly generalizing the definition from (^{3}), we shall call a system (e_k) of elements of a Banach space E orthonormal if the elements of this system are mutually orthogonal: $e_i \perp e_k$ for $i \neq k$, and $\|e_k\| = 1$ for $k = 1, 2, \dots$. If, moreover, the system (e_k) forms a basis in the closure $[e_k]$ of its linear span, then we shall call this basis orthonormal. The notion of an orthogonal basis in Banach spaces in various degrees of generality was introduced in ($^{4-6}$). According to the definition of V. Ya. Kozlov (^{4}), a basis (x_k) is called orthogonal if for all $x \in E$

$$\left\| \sum_{k=1}^{n} f_k(x) x_k \right\| \leq \|x\|, \qquad n = 1, 2, 3, \dots$$

Here (f_k) denotes the system of linear functionals biorthogonal to the basis (x_k). According to James' s definition, a basis (x_k) is called orthogonal if, for any numbers $a_1, a_2, \dots, a_{n+m}$ and all natural n and m,

$$\left\| \sum_{k=1}^{n} a_k x_k \right\| \leq \left\| \sum_{k=1}^{n+m} a_k x_k \right\|.$$

The two definitions indicated are equivalent. From these definitions it follows easily that $x_i \perp x_k$ for $i < k$, $k = 1, 2, \dots$. Mutual orthogonality, however, does not in general follow from these definitions. Moreover, as shown by G. Auerbach ((^{7}), p.205), in every finite-dimensional Banach space there exist orthonormal bases, but even in some three-dimensional spaces, as established by M. Z. Solomyak (^{8}), these bases may fail to be orthogonal in the sense of Kozlov —James.

In the present work we shall, in particular, indicate conditions of smoothness of the unit sphere of a Banach space and of its conjugate which guarantee the basis property of an orthonormal system of elements.

2. Let E be a separable Banach space over the field of real numbers. It is known (^{2,7}) that for the norm $x \mapsto \|x\|$ to be Gâteaux differentiable at a point $x \in E$ in any direction $h \in E$, it is necessary and sufficient that there exist a unique functional $f_x \in S^* = \{f \in E^* \mid \|f\| = 1\}$ such that $f_x(x) = \|x\|$. In this case the variation of the function $\varphi : x \mapsto \|x\|$ at the point x is equal to

$$\delta\varphi(x; h) = \frac{\partial}{\partial t}\bigg|_{t=0} \|x + th\| = \lim_{t\to 0} \frac{\|x + th\| - \|x\|}{t} = f_x(h),$$

and the Gâteaux derivative is $\varphi'(x) = f_x$.

Lemma 1. $x \perp y \Longleftrightarrow f_x(y) = 0$.

Corollary. If in the space E the norm is Gâteaux differentiable, then an orthonormal system is also orthonormal in the sense of Levina and Petunin (^{3}), i.e., each element of this system is orthogonal to the linear span of the remaining elements.

We shall call a Banach space E smooth if the norms in both E and E^* are Gâteaux differentiable. In this case (^{9,10}) the space E is reflexive, and both spaces E and E^* are strictly normed.

Lemma 2. If in a smooth space E the elements e_1 and e_2 are mutually orthogonal and normalized, then the derivative of the function $\varphi : x \mapsto \|x\|$ at the point $\alpha_1 e_1 + \alpha_2 e_2$ $(\alpha_1, \alpha_2 \in R)$ is a linear combination of the derivatives of this function at the points e_1 and e_2, i.e., for any real numbers α_1 and α_2 there are real numbers λ_1 and λ_2 such that

$$f_{\alpha_1 e_1 + \alpha_2 e_2} = \lambda_1 f_{e_1} + \lambda_2 f^*_{e_2}.$$

Proof. Introduce the following notation:

$$E_2 = \{x \in E \mid x = \alpha_1 e_1 + \alpha_2 e_2;\ \alpha_1 \in R;\ \alpha_2 \in R\},$$
$$F_2 = \{y \in E \mid f_{e_1}(y) = f_{e_2}(y) = 0\}.$$

Using the mutual orthogonality of the elements e_1 and e_2, it is easy to show that $E = E_2 \oplus F_2$. From the linear independence of the functionals f_{e_1} and f_{e_2} it follows that

$$E_2^* = \{f \in E^* \mid f = \lambda_1 f_{e_1}|_{E_2} + \lambda_2 f_{e_2}|_{E_2}\},$$

where $f_{e_1}|_{E_2}$ and $f_{e_2}|_{E_2}$ denote the restrictions of the functionals f_{e_1} and f_{e_2} to the subspace E_2. From the smoothness of the subspace E_2 there follows, for the element $x = \alpha_1 e_1 + \alpha_2 e_2 \in E$, the existence of a unique functional

$$f_x = \lambda_1 f_{e_1}|_{E_2} + \lambda_2 f_{e_2}|_{E_2} \in E_2^*$$

such that $\|f_x\|_{E_2^*} = 1, \; f_x(x) = \|x\|$, and the existence of a unique element $x_0 \in E_2, \; \|x_0\| = 1$, such that

$$f_x(x_0) = \|f_x\|_{E_2^*} = 1.$$

Consider the functional

$$f = \lambda_1 f_{e_1} + \lambda_2 f_{e_2}$$

and show that it is an extension (the unique one) of the functional f_x to all of E with preservation of the norm. Indeed, let $z_0 \in E$ be the unique element for which $\|z_0\| = 1$ and $f(z_0) = \|f\|_{E^*}$. If $z_0 = x_0' + y_0$, where $x_0' \in E_2, \; y_0 \in F_2$, then

$$\|f\|_{E^*} = f(z_0) = f(x_0' + y_0) = \lambda_1 f_{e_1}(x_0' + y_0) + \lambda_2 f_{e_2}(x_0' + y_0) =$$
$$= \lambda_1 f_{e_1}(x_0') + \lambda_2 f_{e_2}(x_0') = f(x_0').$$

Consequently, $x_0' = z_0; \; y_0 = 0$. Taking into account that

$$f|_{E_2} = \lambda_1 f_{e_1}|_{E_2} + \lambda_2 f_{e_2}|_{E_2},$$

we have

$$\|f\|_{E^*} = \|f_x\|_{E_2^*} = f_x(x_0') = 1.$$

Consequently, $x_0' = x_0$ and

$$f(x) = f_x(x) = \|x\|.$$

Therefore

$$f_x = f_{\alpha_1 e_1 + \alpha_2 e_2} = f = \lambda_1 f_{e_1} + \lambda_2 f_{e_2}.$$

Lemma 3. If the system $(e_1, e_2, \ldots, e_n)$ of elements of a smooth Banach space E is orthonormal, then for any point $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in R^n$ there exists, and moreover is unique, a point $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in R^n$ such that

$$f_{\alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n} = \lambda_1 f_{e_1} + \lambda_2 f_{e_2} + \ldots + \lambda_n f_{e_n}.$$

Proof. For $n = 2$ the lemma has been proved. Suppose that it is valid for $n = k$. Then, taking into account that by assumption the elements e_{k+1} and $\sum_{i=1}^{k} \alpha_i e_i$ are mutually orthogonal, we have

$$f_{\alpha_1 e_1 + \ldots + \alpha_k e_k + \alpha_{k+1} e_{k+1}} = f_{\|\alpha_1 e_1 + \ldots + \alpha_k e_k\| \frac{\alpha_1 e_1 + \ldots + \alpha_k e_k}{\|\alpha_1 e_1 + \ldots + \alpha_k e_k\|} + \alpha_{k+1} e_{k+1}} =$$

$$= \lambda f_{a_1e_1+\ldots+a_ke_k} + \lambda_{k+1} f_{e_{k+1}} = \lambda\mu_1 f_{e_1} + \ldots + \lambda\mu_k f_{e_k} + \lambda_{k+1} f_{e_{k+1}} = \lambda_1 f_{e_1} + \ldots + \lambda_k f_{e_k} + \lambda_{k+1} f_{e_{k+1}}.$$

Theorem. If E is a smooth Banach space and (e_k) is an orthonormal system in this space, then it forms an unconditional basis in the closure $[e_k]$ of its linear span. This basis is strictly orthogonal in the sense of I. Singer, and hence, a fortiori, in the sense of Kozlov–James.

Proof. According to I. Singer' s definition, it is required to prove that for any two disjoint sets $\{p_i\}_1^n$ and $\{q_j\}_1^m$ of natural numbers and arbitrary real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_m$,

$$\left\| \sum_{k=1}^{n} \alpha_k e_{p_k} + \sum_{j=1}^{m} \beta_j e_{q_j} \right\| \geq \left\| \sum_{k=1}^{n} \alpha_k e_{p_k} \right\|,$$

and moreover the equality sign holds only when $\beta_1 = \beta_2 = \ldots = \beta_m = 0$.

Let $\{p_i\}_1^n \subset N$, $\{q_j\}_1^m \subset N$, and $\{p_i\} \cap \{q_j\} = \emptyset$, and let $\alpha_1, \alpha_2, \ldots, \alpha_n$, $\beta_1, \beta_2, \ldots, \beta_m$ be arbitrary real numbers. By Lemma 3, to the point $(a_1, a_2, \ldots, a_n) \in R^n$ there corresponds a unique point $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in R^n$ such that

$$f_{a_1e_{p_1}+a_2e_{p_2}+\ldots+a_ne_{p_n}} = \lambda_1 f_{e_{p_1}} + \lambda_2 f_{e_{p_2}} + \ldots + \lambda_n f_{e_{p_n}}, \qquad \left\| \sum_{k=1}^{n} \lambda_k f_{e_{p_k}} \right\|_{E^*} = 1.$$

Then

$$\begin{aligned}\left\| \sum_{k=1}^{n} \alpha_k e_{p_k} \right\| &= f_{\sum_{k=1}^{n} \alpha_k e_{p_k}} \left(\sum_{k=1}^{n} \alpha_k e_{p_k} \right) = \sum_{k=1}^{n} \lambda_k f_{e_{p_k}} \left(\sum_{k=1}^{n} \alpha_k e_{p_k} \right) \\ &= \sum_{k=1}^{n} \lambda_k f_{e_{p_k}} \left(\sum_{k=1}^{n} \alpha_k e_{p_k} + \sum_{j=1}^{m} \beta_j e_{q_j} \right) \\ &\leq \left\| \sum_{k=1}^{n} \lambda_k f_{e_{p_k}} \right\|_{E^*} \left\| \sum_{k=1}^{n} \alpha_k e_{p_k} + \sum_{j=1}^{m} \beta_j e_{q_j} \right\|_{E} = \left\| \sum_{k=1}^{n} \alpha_k e_{p_k} + \sum_{j=1}^{m} \beta_j e_{q_j} \right\|.\end{aligned}$$

Moreover, it is clear that the equality sign holds only in the case when $\beta_1 = \beta_2 = \ldots = \beta_m = 0$. The theorem is proved.

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* In the terminology of M. Z. Solomyak, the elements e_1 and e_2 are "linked" (^{8}).

Corollary 1. In every smooth Banach space there exists an infinite-dimensional subspace with an orthonormal basis, and hence also an unconditional basis, strictly orthogonal in the sense of I. Singer.

The assertion follows from the theorem of A. Yu. Levin and Yu. I. Petunin (^{3}) on the existence in every Banach space of an infinite orthonormal system. Corollary 1 partially solves the problem posed by C. Bessaga and A. Pełczyński (^{11}), on the existence in every Banach space of an unconditional basic system.

Corollary 2. The trigonometric system is not orthonormal in $L^p([0, 2\pi])$ for any p.

The assertion follows from the fact established by M. Z. Solomyak (^{8}) that the trigonometric system does not form an orthogonal basis in the sense of Kozlov —James in $L^p([0; 2\pi])$ for any p.

3. Let the smooth space E be densely embedded in some Hilbert space H: $E \neq H$ and $\|x\|_H \leq \|x\|_E$ for $x \in E$. After the natural identifications one may assume that $E \subset H \subset E^*$. Let the system $(e_k) \subset E$ be orthonormal in H.

Proposition 1. In order that, under the indicated conditions, the system $(e_k/\|e_k\|_E)$ be orthonormalized in E, it is necessary and sufficient that the relations

$$\|e_k\|_E \|e_k\|_{E^*} = 1, \qquad k = 1, 2, \dots \tag{1}$$

hold.

Necessity. If the system of elements $x_k = e_k/\|e_k\|_E$, $k = 1, 2, \dots$, is orthonormalized in E, then the functionals biorthogonal to them are the functionals $f_k = \|e_k\|_E e_k$, $k = 1, 2, \dots$, and, since $\|f_k\|_{E^*} = 1$, we have $\|e_k\|_E \|e_k\|_{E^*} = 1$, $k = 1, 2, \dots$

Sufficiency. Under conditions (1),

$$\|f_k\|_{E^*} = \|e_k\|_E \|e_k\|_{E^*} = 1, \qquad k = 1, 2, \dots,$$

and since $\|x_k\|_E = 1$, $k = 1, 2, \dots$, the system (x_k) is orthonormalized in E.

Example 1. The Haar system, orthonormalized in $L^2([0, 1])$, is orthonormalized in $L^p([0, 1])$ for any $p > 1$.

$$e_1(t) \equiv 1; \qquad e_{2^n+k}(t) = \begin{cases} \sqrt{2^n}, & \text{if } t \in [(2k-2)/2^{n+1},\ (2k-1)/2^{n+1}), \\ -\sqrt{2^n}, & \text{if } t \in [(2k-1)/2^{n+1},\ 2k/2^{n+1}], \\ 0, & \text{for the remaining } t \in [0, 1]. \end{cases}$$

It is easy to see that for all p, n, and k,

$$\|e_{2^n+k}\|_p \|e_{2^n+k}\|_{p'} = 1.$$

Example 2. The Rademacher system (r_k), orthonormalized in $L^2([0, 1])$, is orthonormalized in every $L^p([0, 1])$.

Proposition 2. Let (e_k) be an orthonormalized system in E. In order that the system $x_k = e_k/\|e_k\|_H$ be orthonormalized also in H, it is necessary and sufficient that

$$\|e_k\|_H^2 = \|e_k\|_{E^*}, \qquad k = 1, 2, \ldots$$

Necessity. If the system (x_k) is orthonormalized in H, then, according to Proposition 1,

$$1 = \|x_k\|_E \|x_k\|_{E^*} = \left\| \frac{e_k}{\|e_k\|_H} \right\| \left\| \frac{e_k}{\|e_k\|_H} \right\|_{E^*} = \frac{\|e_k\|_{E^*}}{\|e_k\|_H^2},$$

and this means that

$$\|e_k\|_H^2 = \|e_k\|_{E^*}, \qquad k = 1, 2, \ldots .$$

Sufficiency. From the conditions $\|e_k\|_H^2 = \|e_k\|_{E^*}$, $k = 1, 2, \ldots$, it follows that $\|e_k/\|e_k\|_H^2\|_{E^*} = 1$, and the functionals $f_k \in E^*$, $k = 1, 2, \ldots$, generated by the elements $y_k = e_k/\|e_k\|_H^2$: $f_k(x) = (x, y_k)$, satisfy the relations $\|f_k\|_{E^*} = 1$, $f_k(e_k) = 1$. But then $f_k = l_k$, $k = 1, 2, \ldots$, and therefore, for $i \neq k$, $0 = f_{e_i}(e_k) = (e_k, y_i) = (e_k, e_i/\|e_i\|_H^2)$. The obtained orthogonality relations complete the proof.

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Note: Figure translations are in progress. See original paper for figures.

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