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**Abstract**

**Full Text**

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**MATHEMATICS**

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## **ON THE PROBLEM OF CLASSIFYING FILTERS**

*(Presented by Academician P. S. Aleksandrov on 29 VII 1969)*

As usual, by a filter one means a subalgebra of the algebra  $P(A)$  of all subsets of some set  $A$ , with the operations  $\cap$  and  $\cup$ , containing, together with each set belonging to it, every set containing it; moreover, the filter is called proper free if it has no minimal element, and an ultrafilter if it is contained in no proper free filter distinct from itself. Being, undoubtedly, one of the fundamental notions, the concept of a filter is widely used; hence follows the importance of the problem of classifying filters and ultrafilters.

Let us turn first of all to the concept of equivalence of filters. First, regarding a filter as an algebra, one may understand equivalence of filters as an isomorphism of algebras. On the other hand, from the set-theoretic point of view it is natural to understand equivalence of filters as the possibility of establishing not an arbitrary isomorphism between them, but an isomorphism generated by a one-to-one mapping of the sets from whose subsets these filters are composed. The proof of the equivalence of these two notions of equivalence constitutes the content of Theorem 1.

Besides singling out algebraic and cardinal invariants of a filter (in the sense of the notions of equivalence set forth above), their classification by characteristics of one or another topological object uniquely associated with equivalence classes of filters, or else with some set of these classes, is natural. Thus, for example, to each filter one may associate the so-called space associated with the filter (see <sup>(1)</sup>, Ch. I, § 6). It is not hard to verify that homeomorphism of the associated spaces is equivalent to equivalence of filters. Further, using the fact that each filter of subsets of a set  $A$  uniquely determines a closed subset of the Stone-Čech compactification  $\beta A$  of the set  $A$ , regarded as a space with the discrete topology, and considering the possibility of generating a homeomorphism of two closed subsets of  $\beta A$  by a homeomorphism of  $\beta A$  onto itself, we can again obtain certain criteria for equivalence of filters.

The purpose of the present work is to investigate the possibility of classifying filters by means of the method of assigning uniquely to each free filter a certain

topological group, set forth and applied by us in papers <sup>(5,6)</sup>, and to consider one class of ultrafilters distinguished in this way.

**Theorem 1.** *If  $\varphi : \mathfrak{F}_1 \rightarrow \mathfrak{F}_2$  is an isomorphism of free filters  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$  of subsets of the sets  $A_1$  and  $B_1$ , respectively, then there exists, and moreover exactly one, one-to-one mapping  $f : A_1 \rightarrow B_1$  of the set  $A_1$  onto the set  $B_1$  such that  $\varphi(A) = f(A)$  for every  $A \in \mathfrak{F}_1$ .*

**Proof.** To each element  $\alpha \in A_1$  assign the set

$$A(\alpha) = A_1 \setminus \{\alpha\} \subset A_1.$$

By the definition of a free filter,  $A(\alpha) \in \mathfrak{F}_1$  for every  $\alpha \in A_1$ , and one may consider the sets  $B(\alpha) = \varphi(A(\alpha))$ . We shall prove that

$$|B_1 \setminus B(\alpha)| = 1.$$

Indeed, since  $\varphi$  is an isomorphism and  $A(\alpha) \neq A_1$ , we have  $B(\alpha) \neq B_1$ , i.e.

$$|B_1 \setminus B(\alpha)| \geq 1.$$

On the other hand, if

$$|B_1 \setminus B(\alpha)| > 1,$$

then there is a  $\beta \in B_1$  such that

$$B(\alpha) \neq B' = B(\alpha) \cup \{\beta\} \neq B_1.$$

But, by the definition of a filter,  $B' \in \mathfrak{F}_2$ , and by the property of  $\varphi$  then

$$\varphi^{-1}(B(\alpha)) = A(\alpha) \neq \varphi^{-1}(B') \neq \varphi^{-1}(B_1) = A_1,$$

and

$$A(\alpha) \neq A(\alpha) \cup \varphi^{-1}(B') \neq A_1,$$

which cannot be by the definition of the set  $A(\alpha)$ . Thus,

$|B_1 \setminus B(a)| = 1$  for every  $a \in A$ , and one may put  $\psi(a) \in B_1$  so that  $\{\psi(a)\} = B_1 \setminus B(a)$ . Since  $\varphi$  is an isomorphism of filters and hence a monomorphism, if  $a_1 \neq a_2$ , then  $A(a_1) \neq A(a_2)$ ,  $\varphi(A(a_1)) = B(a_1) \neq B(a_2) = \varphi(A(a_2))$ , i.e.  $\psi$  is also a monomorphism and, as is not difficult to verify, an epimorphism. To prove the theorem it now remains only to show that  $\varphi(A) = \psi(A)$  for every  $A \in \mathfrak{F}_1$ . But, taking the definitions into account, it is easy to verify the validity of the following chain of equivalences:  $a \in A \Leftrightarrow A(a) \cup A = A_1 \Leftrightarrow \varphi(A(a)) \cup \varphi(A) = B_1 \Leftrightarrow B(a) \cup \varphi(A) = B_1 \Leftrightarrow (B_1 \setminus \{\psi(a)\}) \cup \varphi(A) = B_1 \Leftrightarrow \psi(a) \in \varphi(A)$ , i.e.  $a \in A \Leftrightarrow \psi(a) \in \varphi(A)$ , which means precisely the coincidence of the images  $\psi(A)$  and  $\varphi(A)$ . The theorem is proved.

**Lemma 1.** *If  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$  are ultrafilters of subsets of the sets  $A_1$  and  $B_1$ , respectively,  $|A_1| = |B_1|$ , and for some subsets  $A \in \mathfrak{F}_1$  and  $B \in \mathfrak{F}_2$  the filters*

$\mathfrak{F}'_1 = \mathfrak{F}_1 \cap P(A)$  and  $\mathfrak{F}'_2 = \mathfrak{F}_2 \cap P(B)$  are equivalent, then the ultrafilters  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$  are also equivalent.

**Lemma 2.** Let  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$  be ultrafilters of subsets of the sets  $A_1$  and  $B_1$ , respectively,  $|A_1| = |B_1|$ , and let  $\varphi : A_1 \rightarrow B_1$  and  $\psi : B_1 \rightarrow A_1$  be multivalued mappings satisfying the following conditions:

- 1°.  $\mathfrak{B}_2 = E\{\varphi(A) \mid A \in \mathfrak{F}_1\}$  is a base of the ultrafilter  $\mathfrak{F}_2$ .
- 2°.  $\mathfrak{B}_1 = E\{\psi(B) \mid B \in \mathfrak{F}_2\}$  is a base of the ultrafilter  $\mathfrak{F}_1$ .
- 3°.  $\varphi(a) \cap E\{\beta \in B_1 \mid a \in \psi(\beta)\} \neq \emptyset$  for every  $a \in A_1$ .

Then the ultrafilters  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$  are equivalent.

**Proof.** Choose arbitrarily, for each  $a \in A_1$ , an element  $\chi(a) \in \varphi(a)$  such that  $\psi\chi(a) \ni a$ , which can be done by virtue of 3°, and show that  $\chi(A_1) = B' = E\{\chi(a) \mid a \in A_1\} \in \mathfrak{F}_2$ . Indeed, if  $A_1 \supset A \neq \emptyset$ , then  $B' \supset \chi(A) \neq \emptyset$ ; by definition  $\chi(A) \subset \varphi(A)$ , and therefore  $\varphi(A) \cap B' \supset \chi(A) \cap B' = \chi(A) \neq \emptyset$ . Hence it follows that the intersection of  $B'$  with every element of the base  $\mathfrak{B}_2$  is nonempty, from which, by the maximality of the ultrafilter  $\mathfrak{F}_2$ , it follows that  $B' \in \mathfrak{F}_2$ . Further, according to the axiom of choice, there exists a set  $A' \subset A_1$  such that the mapping  $\vartheta = \chi|_{A'}$  is one-to-one and  $\vartheta(A') = B'$ . We prove that  $A' \in \mathfrak{F}_1$ . Let  $\beta \in B \subset B'$ . Then  $\vartheta^{-1}(\beta) \in A'$ ,  $\chi\vartheta^{-1}(\beta) = \beta$ , and therefore  $\psi(\beta) = \psi\chi\vartheta^{-1}(\beta) \ni \vartheta^{-1}(\beta)$  by the definition of  $\chi$ , whence it follows that if  $B' \supset B \neq \emptyset$ , then  $\psi(B) \supset \vartheta^{-1}(B) \neq \emptyset$  for every  $B \in P(B') \cap \mathfrak{F}_2$ . But  $P(B') \cap \mathfrak{F}_2$  is a base of the ultrafilter  $\mathfrak{F}_2$ , since  $B' \in \mathfrak{F}_2$ , and then  $E\{\psi(B) \mid B \in P(B') \cap \mathfrak{F}_2\}$  is a base of the ultrafilter  $\mathfrak{F}_1$ ; and by the maximality of  $\mathfrak{F}_1$  as an ultrafilter, again  $A' \in \mathfrak{F}_1$ . Thus,  $\vartheta : A' \rightarrow B'$  is a one-to-one mapping,  $A' \in \mathfrak{F}_1$ ,  $B' \in \mathfrak{F}_2$ ,  $\vartheta(A) \subset \varphi(A)$  for every  $A \subset A'$  (since, as we have shown,  $\chi(A) \subset \varphi(A)$  and, evidently,  $\vartheta(A) = \chi(A)$ ) and  $\vartheta^{-1}(B) \subset \psi(B)$  for every  $B \subset B'$ . We now show that the filters  $\mathfrak{F}'_1 = \mathfrak{F}_1 \cap P(A')$  and  $\mathfrak{F}'_2 = \mathfrak{F}_2 \cap P(B')$  are equivalent. Indeed, the image of a filter under a one-to-one mapping is again a filter, and since  $\vartheta(A) \subset \varphi(A)$ , each element of the base  $E\{\varphi(A) \mid A \in P(A') \cap \mathfrak{F}_2\}$  of the filter  $\mathfrak{F}'_2$  contains an element  $\vartheta(A)$  of the filter  $\vartheta(\mathfrak{F}'_1) = E\{\vartheta(A) \mid A \in \mathfrak{F}'_1\}$ ; hence, by the maximality of  $\mathfrak{F}'_2$  as an ultrafilter, the filters  $\mathfrak{F}'_2$  and  $\vartheta(\mathfrak{F}'_1)$  coincide. Analogously the converse is true, i.e. the filter  $\vartheta^{-1}(\mathfrak{F}'_2) = E\{\vartheta^{-1}(B) \mid B \in \mathfrak{F}'_2\}$  is a base of the ultrafilter  $\mathfrak{F}'_1$ , whence again  $\mathfrak{F}'_1 = \vartheta^{-1}(\mathfrak{F}'_2)$ . Thus we have shown that under the mapping  $\vartheta$  the image (preimage) of every element of the filter  $\mathfrak{F}'_1$  (respectively  $\mathfrak{F}'_2$ ) is an element of the filter  $\mathfrak{F}'_2$  (respectively  $\mathfrak{F}'_1$ ), and the filters  $\mathfrak{F}'_1 = \mathfrak{F}_1 \cap P(A')$  and  $\mathfrak{F}'_2 = \mathfrak{F}_2 \cap P(B')$  are equivalent by definition; hence by Lemma 1 the equivalence of the filters  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$  follows. The lemma is proved.

In the papers <sup>(5, 6)</sup> we set out the construction of a topology  $\mathcal{T}(\mathfrak{F})$  on the product  $\prod\{X_\alpha \mid \alpha \in A\}$  and on  $\mathfrak{S}\{X_\alpha \mid \alpha \in A\}$  of topological spaces, which is induced by a certain free filter  $\mathfrak{F}$  on the set of indices  $A$  and by the topology of the spaces  $X_\alpha$  themselves. The topology  $\mathcal{T}(\mathfrak{F})$  gives rise, in the case of topological groups, to a new topological group (in particular, Ticho-

Novskaya and the so-called strong (box) topologies are also topologies of the form  $\mathcal{T}(\mathfrak{F})$  for a suitable choice of the filter  $\mathfrak{F}$ .

For the case of  $\mathfrak{S}$ -products of cyclic discrete groups one can give a more algebraic definition of the topology  $\mathcal{T}(\mathfrak{F})$ , which we shall use below.

If  $X$  is a subset of a group  $G$ , then by  $[X]$  we denote the minimal nonempty subgroup of the group  $G$  containing the set  $X$ .

Let  $G$  be an Abelian group and  $X_1 \subset G$  its basis, i.e., such a system of generators that from the equality  $n_1g_1 + n_2g_2 + \dots + n_kg_k = 0$ , where  $n_i \in N$  and  $g_i \neq g_j \in X_1$  for  $i \neq j$ , it follows that  $n_1g_1 = n_2g_2 = \dots = n_kg_k = 0$  (see, for example, <sup>(2)</sup>, § D18), and let  $\mathfrak{F}$  be a free filter of subsets of the set  $X_1$ . Then, as is not hard to verify, the system  $\mathfrak{B} = E\{[X] \mid X \in \mathfrak{F}\}$  satisfies all the axioms of a system of neighborhoods of zero of a topological group (see, for example, <sup>(3)</sup>, § 18) and thereby defines on  $G$  a topology turning  $G$  into a topological group, nondiscrete if the filter  $\mathfrak{F}$  is proper.

**Theorem 2.** *Let  $G$  and  $H$  be some Abelian groups,  $X_1$  and  $Y_1$  their bases, and  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$  ultrafilters of subsets of the sets  $X_1$  and  $Y_1$ , respectively. Then from an isomorphism of topological groups:  $G$  in the topology  $\mathcal{T}(\mathfrak{F}_1)$  and  $H$  in the topology  $\mathcal{T}(\mathfrak{F}_2)$ , it follows that the ultrafilters  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$  are equivalent.*

**Proof.** Directly from the definition of a basis of a group it follows that for every nonzero element  $g$  of the group  $G$  there is uniquely determined a finite set  $k_1(g) \subset X_1$  such that  $g = \sum\{n(g, x)x \mid x \in k_1(g)\}$ , where  $n(g, x)x \neq 0$ , and, moreover, if  $g \in [X]$  for some  $X \subset X_1$ , then necessarily  $k_1(g) \subset X$ . Similarly we define  $k_2(h) \subset Y_1$  for every  $h \in H \setminus \{0\}$ , and put  $\varphi(x) = k_2(f(x))$ ,  $\psi(y) = k_1(f^{-1}(y))$ , where  $f$  is an isomorphism of the groups (topological)  $G$  and  $H$ .

Let us now prove that the multivalued mappings  $\varphi$  and  $\psi$  satisfy conditions 1°, 2°, and 3° of Lemma 2. Let  $Y' \in \mathfrak{F}_2$ . Then  $[Y']$  is a neighborhood of zero of the group  $H$  by the definition of the topology  $\mathcal{T}(\mathfrak{F}_2)$ , and since  $f$  is an isomorphism of topological groups and  $E\{[X] \mid X \in \mathfrak{F}_1\}$  is a base at zero in  $G$ , we have  $f([X']) \subset [Y']$  for some  $X' \in \mathfrak{F}_1$ ; hence  $f(x) \in [Y']$  for every  $x \in X'$ , and then, by definition,  $Y' \supset k_2(f(x)) = \varphi(x)$ , whence  $\varphi(X') = \bigcup\{k_2(f(x)) \mid x \in X'\} \subset Y'$ , i.e.  $E\{\varphi(X) \mid X \in \mathfrak{F}_1\}$  is a base of the ultrafilter  $\mathfrak{F}_2$ , which means that condition 1° of Lemma 2 is fulfilled. From the symmetry of the conditions of the theorem with respect to  $G$  and  $H$  it is clear that the fulfillment of condition 2° is proved analogously.

Let us show the fulfillment of condition 3°. According to the definitions of  $\varphi$  and  $\psi$ , and using the fact that  $f$  is an isomorphism, for arbitrary  $x_0 \in X_1$  and  $y \in Y_1$  we have

$$f(x_0) = \sum\{n(f(x_0), y)y \mid y \in \varphi(x_0)\}, \quad f^{-1}(y) = \sum\{n(f^{-1}(y), x)x \mid x \in \psi(y)\},$$

whence

$$x_0 = \sum \{ n(f(x_0), y) f^{-1}(y) \mid y \in \varphi(x_0) \},$$

i.e.

$$x_0 = \sum \sum \{ n(f(x_0), y) \cdot n(f^{-1}(y), x) x \mid y \in \varphi(x_0) \ \& \ x \in \psi(y) \}.$$

Since on the left  $x_0$  stands with a nonzero coefficient, by the definition of a basis it must also be present in the sum on the right, i.e.  $x_0 \in \psi(y_0)$  for some  $y_0 \in \varphi(x_0)$ , whence  $y_0 \in \varphi(x_0) \cap E\{y \in Y_1 \mid x_0 \in \psi(y)\} \neq \emptyset$ , which means the fulfillment of condition 3° of Lemma 2; from this the assertion of the theorem follows. The theorem is proved.

Apparently, the most convenient for the study of filters by means of considering the  $(\mathfrak{S})$ -product in the topology  $\mathcal{T}(\mathfrak{F})$  is the  $(\mathfrak{S})$ -product of discrete two-element groups, as a sufficiently well surveyed group. Moreover, only in this case is it possible to study the filter from the point of view of the extreme disconnectedness of the  $\mathfrak{S}$ -product in the topology  $\mathcal{T}(\mathfrak{F})$ , since the following holds.

**Theorem 3.** *If  $\mathfrak{F}$  is a filter of subsets of the index set  $A_1$*

*of families of topological spaces  $E\{X_\alpha \mid \alpha \in A_1\}$  and  $\mathfrak{S}\{X_\alpha \mid \alpha \in A_1\}$  in the topology  $\mathcal{T}(\mathfrak{F})$  are extremely disconnected, then for some  $A \in \mathfrak{F}$  each of the spaces  $X_\alpha$  contains no more than two points for every  $\alpha \in A$ .*

We shall denote by  $\bar{P}$  the set of all finite subsets of the corresponding set.

**Definition 1.** A filter  $\mathfrak{F}$  of subsets of the set  $A_1$  is called a *k-filter* if it satisfies the following two conditions:

- A) If  $A \in \mathfrak{F}$  and  $\varphi : A \rightarrow \mathfrak{F}$  is an arbitrary single-valued mapping, then there exists  $A' \in \mathfrak{F} \cap \bar{P}(A)$  such that for each  $M \in \bar{P}(A) \setminus \{\emptyset\}$  one can choose  $\alpha \in M$  for which  $M \subset \{\alpha\} \cup \varphi(\alpha)$ .
- B) If  $\mathfrak{M} \subset \bar{P}(A_1)$  is such that  $\emptyset \notin \mathfrak{M}$  and for each  $M \in \bar{P}(A_1) \setminus \mathfrak{M}$  there is an  $A(M, \mathfrak{M}, \mathfrak{F}) \in \mathfrak{F}$  such that  $M \cup \{\alpha\} \in \mathfrak{M}$  for every  $\alpha \in A(M, \mathfrak{M}, \mathfrak{F})$ , then  $\mathfrak{M} \cap \bar{P}(A) = \emptyset$  for some  $A \in \mathfrak{F}$ .

For countable  $A_1$  this definition was given by us in <sup>6</sup>, and there, on the basis of this definition, it was proved that if  $\mathfrak{F}$  is a *k-ultrafilter*, then the  $\mathfrak{S}$ -product of discrete two-point spaces in the topology  $\mathcal{T}(\mathfrak{F})$  is extremely disconnected.

**Theorem 4.** *If an ultrafilter of subsets of a set  $A$  satisfies condition A) of the definition of a k-ultrafilter, then it also satisfies the following condition:*

- C) *If  $\mathfrak{D}$  is some partition of the set  $A$  into disjoint subsets, then either  $\mathfrak{D} \cap \mathfrak{F} \neq \emptyset$ , or there exists  $A \in \mathfrak{F}$  such that  $|A \cap D| < 2$  for every  $D \in \mathfrak{D}$ .*

Choquet in <sup>7</sup> calls filters satisfying condition C) absolute; filters satisfying condition C) with the weakening replacement in the inequality of two by  $\aleph_0$ , and introduced by Rudin in <sup>4</sup>, are  $\delta$ -stable. In <sup>5</sup> we showed that every  $k$ -ultrafilter is  $\delta$ -stable, which also follows from Theorem 4; however, the question of the coincidence of the sets of  $k$ -ultrafilters and absolute filters remains open (inclusion in one direction is obvious).

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