ON k-DECOMPOSABLE SPACES THAT ARE NOT MAXIMALLY DECOMPOSABLE

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Abstract

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By a space we shall always mean a T_0-space dense in itself.

I. Theorem 1. Every space can be represented as the disjoint union of a closed k-decomposable* subspace of its own and an open subset of its own containing no nonempty k-decomposable subspace, where $k \geq 1$.

For the case $k = 2$ this theorem was proved by Hewitt ([7]).

Definition 1. A space X is called a **filter space** if its topology is $\mathfrak{F} \cup \{\Lambda\}$, where $\mathfrak{F}$ is a filter in the set X (if $\mathfrak{F}$ is a filter in the set X, then $\mathfrak{F} \cup \{\Lambda\}$ satisfies all the axioms of a topology in the set X).

Lemma 1. A filter space that is the intersection of k ultrafilters, $k \geq 1$, is not k^+-decomposable.

Lemma 2. If some end** of the space X forms a base of a filter in the set X that is the intersection of k ultrafilters, $k \geq 1$, then the space X is not k^+-decomposable.

Theorem 2. A space X is not $(k+1)$-decomposable, $1 \leq k < \aleph_0$, if and only if there exists an end of the space X forming a base of a filter in the set X that is the intersection of k ultrafilters.

For the case $k = 1$ this theorem was proved in ([3]).

Theorem 3. Let $1 \leq k < \aleph_0$. Then the following assertions concerning a dense-in-itself T_1-space X are equivalent:

1) X is k-decomposable and contains no nonempty $(k+1)$-decomposable subspace;
2) X can be represented as the union of k pairwise disjoint everywhere dense CI-subspaces***;
3) X is k-decomposable and can be represented as the union of k of its own SI-subspaces.

Theorem 4. Let $1 \leq k < \aleph_0$. Then the following assertions concerning a nonempty dense-in-itself T_1-space X are equivalent:

1) X contains an open everywhere dense set that is the union of k of its own SI-subspaces;

2) every end of the space X forms a base of a filter in the set X that is the intersection of k ultrafilters;

3) for every point of some set dense in X there exists an end of the space X containing the system of all neighborhoods of this point and forming a base of a filter in the set X that is the intersection of k ultrafilters;

4) X contains no nonempty open $(k+1)$-decomposable subspace.

For the case $k = 1$ this theorem was proved in ([3]).

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* A space is called k-decomposable, where k is a cardinal number ≥ 1, if it contains (or, equivalently, can be represented as the union of) k pairwise disjoint sets dense in it. A 2-decomposable space is called decomposable; a ΔX-decomposable space is maximally decomposable (ΔX-dispersion character of the space X is the minimum cardinality of a nonempty open set in X).

** An end of the space X is a maximal centered system of open subsets of X. For properties of ends see the works of P. S. Aleksandrov ([1]), C. Iliadis and C. Fomin ([5]).

*** A space is called an SI-space, or simply SI, if it is dense in itself and contains no nonempty decomposable subspace.

II. Let $X = \bigcup\{X_\lambda : \lambda \in L\}$, where X is a space and $\{X_\lambda : \lambda \in L\}$ is a family of subsets dense in X. A set $A \subseteq X$ will be called *marked* by the representation $X = \bigcup\{X_\lambda : \lambda \in L\}$ of the space X if A is dense in itself and if $A \subseteq [A \cap X_\lambda]$ for every $\lambda \in L$. A system of subsets of the space X, for which the above representation of cardinality $|L| = k \geqslant 1$ is given, will be called a q_k-system (respectively, a d_k-system), relative to the given representation, if the intersection of every finite family of sets that are elements of this system is marked (respectively, everywhere dense and marked) by the given representation of the space X or is empty. A maximal centered q_k-system (respectively, d_k-system) will be called a q_k-end (respectively, a d_k-end).

A space Y is called a (proper) relaxation of a space X (an expansion in the terminology of Hewitt ([7])) if X and Y are defined on one and the same set of points and if the topology of the space Y (strictly) contains the topology of the space X. If $\mathcal{A}$ is some system of subsets of the space X with topology $\mathcal{T}$, then the set X, endowed with the topology with prebase $\mathcal{T} \cup \mathcal{A}$, is a relaxation of the space X, which we shall call an $\mathcal{A}$-relaxation of the space X. The topology of the $\mathcal{A}$-relaxation of the space X is, obviously, the least upper bound of two topologies—the topology $\mathcal{T}$ of the space X and the topology with prebase $\mathcal{A}$.

Definition 2. A space X is called an $M'_k I$-space if X can be represented as the union of $k \geqslant 1$ subsets dense in it in such a way that every everywhere dense set marked by this representation is open in X. Any such representation will be called a representation of the given $M'_k I$-space. An $M'_k I$-space with a disjoint representation will be called an M_{kI}-space.

Obviously, $M_1 I$-spaces are MI-spaces.

Theorem 5. Every nonempty space $X = \bigcup\{X_\lambda : \lambda \in L\}$, where X_λ is dense in X, $\lambda \in L$, and $|L| = k \geqslant 1$, has a relaxation Y identically θ-homeomorphic to it that is an $M'_k I$-space with representation $Y = \bigcup\{X_\lambda : \lambda \in L\}$. Every such relaxation is connected (extremally, H-closed, Hausdorff, Urysohn) if and only if the space X is connected (respectively extremally, respectively H-closed, respectively Hausdorff, respectively Urysohn). If this relaxation is proper, then it is nonsemiregular. Such relaxations are precisely all $\mathcal{A}$-relaxations of the space X, where $\mathcal{A}$ runs through the set of all d_k-ends of the space X (relative to the given representation). Moreover, every d_k-end of the space X is the system of all everywhere dense marked sets of the corresponding relaxation. If, in addition, X is a T_1-space, then there exists a free $*$ filter in the set X (every d_k-end in X is such) such that the least upper bound of the space of this filter and of the space X is an $M'_k I$-space.

Theorem 6. Let X be a dense-in-itself T_1-space,

$$X = \bigcup\{X_\lambda : \lambda \in L\}, \quad X_\lambda \text{ dense in } X, \quad \lambda \in L, \quad X_\lambda \cap X_{\lambda'} = \Lambda, \ \text{ if } \lambda \neq \lambda',$$

and $|L| = k \geqslant 1$. Then the following assertions about the space X are equivalent:

1) X is an M_{kI}-space with the given representation;
2) every set $T \subseteq X$ such that $\operatorname{int}_{X_\lambda}(T \cap X_\lambda) = \Lambda$ for every $\lambda \in L$ is discrete in X (i.e. closed in X and discrete in itself);
3) X_λ is an MI for every $\lambda \in L$, and every set $T \subseteq X$ that meets X_λ in a discrete subspace for every $\lambda \in L$ is discrete in X.

Corollary. If $1 \leqslant k < \aleph_0$, then a nonempty M_{kI}-space satisfying the separation axiom T_1 is not $(k+1)$-decomposable.

* A filter is called free if it has empty intersection.

This corollary and Theorem 5 prove Theorem 7 from (4).

Theorem 7. Let $1 \leq k < \aleph_0$. Then the following assertions concerning a k-decomposable T_1-space X are equivalent:

1) X is an $M_k I$-space with respect to some representation of the space X as the union of k pairwise disjoint everywhere dense subsets of it;
2) $X = \bigcup\{X_i : i = 1, \ldots, k\}$, X_i contains an open set SI dense in it, $i = 1, \ldots, k$, and every nowhere dense subset of X is discrete in X;

3) X is an $M_k I$-space with respect to any representation of the space X as the union of k pairwise disjoint everywhere dense subsets of it.

Theorem 8. A space $X = \bigcup\{X_\lambda : \lambda \in L\}$, X_λ dense in X, $\lambda \in L$, $|L| = k \geq 1$, is an $M'_k I$-space with the given representation if and only if every proper refinement Y of the space X has another reserve of canonical open sets, or there exists $\lambda \in L$ such that X_λ is not dense in Y.

III. **Definition 3.** A space X is called an M'_k-space if X can be represented as the union of $k \geq 1$ subsets dense in it in such a way that every set marked by this representation is open in X. An M'_k-space with such a disjoint representation will be called an M_k-space*.

Obviously, M_1-spaces are precisely maximal** spaces. Every M'_k-space is an $M'_k I$-space with respect to the same representation.

A space in which every semi-open*** set is open will be called **global**, and a global Hausdorff space—globally disconnected.

Proposition 1. 1°. Every M'_k-space is global.
2°. Every Hausdorff M'_k-space is Urysohn.

Theorem 9. Every (Hausdorff) space $X = \bigcup\{X_\lambda : \lambda \in L\}$, where X_λ is dense in X, $\lambda \in L$, and $|L| = k \geq 1$, has a (Urysohn) refinement Y which is an M'_k-space with representation $Y = \bigcup\{X_\lambda : \lambda \in L\}$. Such refinements are precisely all $\mathcal{A}$-refinements of the space X, where $\mathcal{A}$ ranges over the set of all maximal q_k-systems of the space X, each of the latter being the topology of the refinement corresponding to it. Such a refinement is also every $\mathcal{A}$-refinement of the space X, where $\mathcal{A}$ is a q_k-end in X. If X is a nonempty self-dense T_1-space, then there exists on the set X at least one anti-Hausdorff T_1-topology (namely, $\mathcal{A} \cup \{\Lambda\}$, where $\mathcal{A}$ is a q_k-end in X) such that the upper bound of this topology and the topology of the space X is the topology of an M'_k-space.

Theorem 10 ([2]). Let X be a self-dense T_1-space,

$$X = \bigcup\{X_\lambda : \lambda \in L\},$$

X_λ dense in X, $\lambda \in L$, $X_\lambda \cap X_{\lambda'} = \Lambda$ if $\lambda \neq \lambda'$, and $|L| = k \geq 1$. In order that X be an M_k-space with the given representation, it is necessary and sufficient that X_λ be a maximal subspace for every $\lambda \in L$ and that every set $T \subset X$ intersecting X in a discrete subspace for every $\lambda \in L$ be discrete in X.

Corollary ([2]). If $1 \leq k < \aleph_0$, then a nonempty M_k-space satisfying the separation axiom T_1 is not $(k+1)$-decomposable.

Theorem 11. Let $1 \leq k < \aleph_0$. Then the following assertions concerning a nonempty k-decomposable T_1-space X are equivalent:

* An equivalent (for $k \geq 2$) definition of an M_k-space was given in ([3]).
** A space is called maximal if it is self-dense and if every self-dense subspace of this space is open.

*** A subset A of a space X is called semi-open if $U \subset A \subset [U]$ for some set U open in X.

1) X is an M_k-space with respect to some representation of the space X in the form of a union of k pairwise disjoint everywhere dense subsets of it;
2) $X = \bigcup\{X_i : i = 1, \dots, k\}$, X_i contains an open set SI dense in it, $i = 1, \dots, k$, and the space X is global;
3) X is an M_k-space with respect to any representation of the space X in the form of a union of k pairwise disjoint everywhere dense subsets of it;
4) X is global and for every point $x \in X$ the system $\{U \setminus \{x\} : U$ is a neighborhood of the point x in $X\}$ forms a base of a filter in the set X, which is the intersection of k ultrafilters.

Theorem 12. Let X be a nonempty m-decomposable T_1-space and $sX \leq m$. Then for every k, $2 \leq k < m$, X has a relaxation Y such that: 1) Y is an M_k-space, 2) $sY = sX$, 3) $\Delta Y \geq m$, 4) Y is not l^+-decomposable, where $l = \max\{k, sX\} < m$.

Corollary. For every cardinal number $k \geq \aleph_0$ there exists a Urysohn (M_k^-) space X, $|X| = \Delta X = 2^{2^k}$, $sX \leq k$, X is k-decomposable, but is not k^+-decomposable and hence is not maximally decomposable. Moreover, there exists a separable Urysohn space Z, $|Z| = \Delta Z = 2^{\aleph}$, Z is $\aleph_0$-decomposable, but is not maximally decomposable.

This corollary gives a positive solution of the problem Q_2 of Ceder and Pearson ([6]) in the class of Urysohn spaces (the existence of infinitely decomposable non-maximally decomposable T_1-spaces was proved in ([3])).

Theorem 13. The space $X = \bigcup\{X_\lambda : \lambda \in L\}$, X_λ is dense in X, $\lambda \in L$, and $|L| = k \geq 1$, is an M_k'-space with the given representation if and only if for every proper dense-in-itself relaxation Y of the space X there exists $\lambda \in L$ such that X_λ is not dense in Y.

For M_k-spaces, $k \geq 2$, this theorem was proved in ([2]).

Corollary 1. A T_1-space X is an M_k-space if and only if X is k-decomposable, and every proper relaxation of the space X is not k-decomposable, where $2 \leq k < \aleph_0$.

Corollary 2. Let $2 \leq k < \aleph_0 \leq n$, where n either is equal to $\aleph_0$, or satisfies the condition $n^{\aleph_0} = n$. Then there exists a k-decomposable Urysohn space of cardinality n, in which every set of cardinality less than n is discrete and every proper relaxation of which is not k-decomposable.

IV. The problem Q_1 of Ceder and Pearson ([6]) is formulated as follows: is the product of two spaces, one of which is maximally decomposable, maximally decomposable?

Theorem 14. If there exists a free countably centered* ultrafilter, then the problem Q_1 has a negative solution: the product of the space of such an ultrafilter by the space of rational numbers is not maximally decomposable.

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* A family of sets is called countably centered if every countable subfamily of it has nonempty intersection.

Note: Figure translations are in progress. See original paper for figures.